

# ESSENTIAL SURFACES IN (3-MANIFOLD, GRAPH) PAIRS AND LEVELING EDGES OF HEEGAARD SPINES

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**ABSTRACT.** Let  $T$  be a graph in a compact, orientable 3-manifold  $M$  and let  $\Gamma$  be a subgraph.  $T$  can be placed in bridge position with respect to a Heegaard surface  $H$ . We use untelescoping and consolidation operations to show that if  $H$  is what we call  $(T, \Gamma)$ -c-weakly reducible in the complement of  $T$  then either the exterior of  $T$  contains an essential meridional surface or one of several “degenerate” situations occurs. This extends previous results of Hayashi-Shimokawa and Tomova to graphs in 3-manifolds which may have non-empty boundary. We apply this result to the study of leveling edges of trivalent Heegaard spines.

## 1. INTRODUCTION

It is a seminal result of Casson and Gordon that if a 3-manifold has a weakly reducible, irreducible Heegaard splitting, then the manifold is Haken [CG]. Strongly irreducible Heegaard surfaces often behave much like incompressible surfaces so this result guarantees that every manifold contains a potentially useful surface.

Casson and Gordon’s result was reproved by Scharlemann and Thompson in [ST1] as a consequence of their work on generalized Heegaard splittings of 3-manifolds. In this construction instead of splitting the manifold along a single surface into two compression bodies, they obtain two collections of surfaces, thin and thick, whose union cuts the manifold into multiple compression bodies. Scharlemann and Thompson define a complexity on such splittings and show that if a generalized Heegaard surface is of minimal complexity, each of the thin surfaces is incompressible.

Because of the fundamental importance of this result, there have been several generalizations. Hayashi and Shimokawa showed in [HS3] that if  $K$  is a properly embedded tangle in a 3-manifold  $M$  that has a weakly reducible bridge surface, then either the bridge surface can be simplified (we will make this precise later) or the tangle complement contains a meridional essential surface. Tomova in [T] weakened the hypothesis and showed that

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a slightly weaker result can be obtained if  $M$  is a closed manifold containing a link and it has a  $c$ -weakly reducible bridge surface, see Section 2.4 for the relevant definitions. However due to technical difficulties the result was not proven for manifolds with boundary.

In their recent paper [TT] the current authors classify all bridge surfaces for  $(N, \tau)$  where  $N$  is a compression body and  $\tau$  is a graph so that  $\partial_+ N$  is parallel to a neighborhood of  $\partial_- N \cup \tau$ . This opens the door for a complete generalization of Casson and Gordon's result. In the current paper we will show that if  $M$  is a compact manifold and  $T$  is a properly embedded graph so that there is a bridge surface for  $(M, T)$  that is  $(T, \Gamma)$ - $c$ -weakly reducible, then either the bridge surface can be simplified or  $M$  contains an essential surface transverse to  $T$ . More precisely we show the following:

**Theorem 7.2.** *Let  $M$  be a compact, orientable 3-manifold containing a properly embedded graph  $T$  and let  $\Gamma$  be a subgraph of  $T$ . Furthermore assume that  $M - T$  is irreducible and no sphere in  $M$  intersects  $T$  exactly once. Let  $H$  be a  $(T, \Gamma)$ - $c$ -weakly reducible Heegaard surface for  $(M, T)$ . Then one of the following holds:*

- (1) *there is a multiple  $\Gamma$ -Heegaard splitting  $\mathcal{H}$  for  $(M, T, \Gamma)$  so that each thin surface is  $T$ -essential and each thick surface is  $(T, \Gamma)$ - $c$ -strongly irreducible in the component of  $M - \mathcal{H}^-$  containing it.*
- (2)  *$H$  contains a generalized stabilization,*
- (3)  *$H$  is a perturbed bridge surface,*
- (4)  *$T$  has a removable path.*

The proof of this theorem is constructive, i.e., if (2), (3), and (4) do not occur, we give an algorithm to construct a multiple  $\Gamma$ -bridge splitting satisfying (1).

In the second part of the paper we give an application of this result to the question of when it is possible to level certain edges in a graph. Known results of this nature include the result by Goda, Scharlemann and Thompson that for tunnel number one knots the unknotting tunnel can be levelled [GST], and the theorem of Scharlemann and Thompson [ST3], stating that if  $T$  is a trivalent genus 2 Heegaard spine for  $S^3$  isotoped to be in thin position with respect to a Heegaard sphere  $H$  for  $S^3$  then  $T$  is in extended bridge position with respect to  $H$  and some interior edge of  $T$  is a perturbed level edge. We show the following partial generalization of the Scharlemann-Thompson result.

**Theorem 9.1.** *Suppose  $H$  is a Heegaard surface for  $M$  and  $T$  is an irreducible trivalent Heegaard spine for a closed manifold  $M$  in minimal bridge position with respect to  $H$ . Then one of the following occurs:*

- (1)  $H$  is stabilized, meridionally stabilized, or bimeridionally stabilized as a splitting of  $(M, T)$ .
- (2)  $T$  has a perturbed level edge.
- (3)  $T$  contains a perturbed level cycle.
- (4) There is an essential meridional surface  $F$  in the exterior of  $T$  such that  $\text{genus}(F) \leq \text{genus}(H)$ .

## 2. DEFINITIONS

**2.1. Surfaces in  $(M, T)$ .** Let  $T$  be a finite graph. Unless otherwise specified we assume that  $T$  has no valence 2 vertices as such vertices can generally be deleted and their adjacent edges amalgamated. We say that  $T$  is *properly embedded* in a 3-manifold  $M$  if  $T \cap \partial M$  is the set of all valence 1 vertices of  $T$ . We will denote the pair  $(M, T)$ .

Suppose that  $F \subset M$  is a surface such that  $\partial F \subset (\partial M \cup T)$ . Then  $F$  is  *$T$ -compressible* if there exists a compressing disk for  $F - T$  in  $M - T$ . If  $F$  is not  $T$ -compressible, it is  *$T$ -incompressible*.  $F$  is  *$T$ - $\partial$ -compressible* if there exists a disk  $D \subset M - T$  with interior disjoint from  $F$  such that  $\partial D$  is the endpoint union of an arc  $\gamma$  in  $F$  and an arc  $\delta$  in  $\partial M$ . We require that  $\gamma$  not be parallel in  $F - T$  to an arc of  $\partial F - T$ . If  $F$  is not  $T$ - $\partial$ -compressible, it is  *$T$ - $\partial$ -incompressible*. Finally suppose  $\Gamma$  is some subgraph of  $T$ . We will say that  $F$  is  *$(T, \Gamma)$ -cut-compressible*, if there exists a compressing disk  $D^c$  for  $F - T$  in  $M$  so that  $|D^c \cap T| = 1$  and that point is contained in  $\Gamma$ . We also require that  $\partial D^c$  is not parallel in  $F - T$  to a puncture  $T \cap F$ . We call  $D^c$  a  *$(T, \Gamma)$ -cut-disk*. A  *$(T, \Gamma)$ -c-disk* will be either a  $T$ -compressing disk or a  $(T, \Gamma)$ -cut-disk. A surface  $F$  in  $M$  is called  *$T$ -parallel* if  $F$  is boundary parallel in  $M - \mathring{\eta}(T)$  and  *$T$ -essential* if it is  $T$ -incompressible and not  $T$ -parallel.

**2.2. Trivially embedded graphs in compression bodies.** Let  $C$  be a compression body and  $T$  be a properly embedded graph in  $C$ . A connected component  $\tau$  of  $T$  is *trivial* in  $C$  if it is one of four types:

- (1) *Bridge arc*: a single edge with both endpoints in  $\partial_+ C$  which is parallel to an arc in  $\partial_+ C$ . The disk of parallelism is called a *bridge disk*.
- (2) *Vertical edge*: a single edge with one endpoint in  $\partial_+ C$  and one endpoint in  $\partial_- C$  that is isotopic to  $\{\text{point}\} \times I$ .
- (3) *Pod*: a graph with a single vertex in the interior of  $C$  and with all valence 1 vertices lying in  $\partial_+ C$  so that there is a disk  $D$  with  $\partial D \subset \partial_+ C$  inessential in  $\partial_+ C$  and so that  $\tau \subset D$ . The disk  $D$  is called a *pod disk*. Each of the components of  $D - \tau$  will be called a *bridge disk* as these components play the same role as bridge disks for bridge arcs.

- (4) *Vertical pod*: a graph with a single vertex in the interior of  $C$  and with one valence 1 vertex lying in  $\partial_- C$  and all other valence 1 vertices in  $\partial_+ C$  so that if the edge adjacent to  $\partial_- C$  is removed the resulting graph is a pod and if instead all but one of the edges adjacent to  $\partial_+ C$  are removed, the result is a vertical edge with a valence 2 vertex in its interior. The edges that have one endpoint in  $\partial_+ C$  are called *pod legs* and the other edge is called a *pod handle*.

If all components of  $T$  are trivially embedded, then we say that  $T$  is *trivially embedded* in  $C$ .

### 2.3. Trivially embedded graphs in $\Gamma$ -compression bodies.

**Definition 2.1.** Let  $C$  be a compression body containing a properly embedded graph  $T$  and let  $\Gamma$  be a subgraph of  $T$ . Suppose that there is a collection of  $(T, \Gamma)$ -cut-disks,  $\mathcal{D}^c$ , such that:

- (1) for each edge of  $\Gamma$  there is at most one disk in  $\mathcal{D}^c$  intersecting it,
- (2) each edge of  $\Gamma$  intersected by  $\mathcal{D}^c$  has both endpoints on  $\partial_- C$ ,
- (3) cut-compressing  $C$  along all cut-disks  $\mathcal{D}^c$  produces a union of compression bodies  $C_1, \dots, C_n$ ,
- (4) for each  $i$  the graph  $C_i \cap T$  is trivially embedded in  $C_i$ .

Then we call the triple of the compression body and the graphs a  $\Gamma$ -compression body containing a trivially embedded graph  $T - \Gamma$  and denote it  $(C, T, \Gamma)$ .

Note that if  $\Gamma = \emptyset$  then  $(C, T, \Gamma) = (C, T)$  is a compression body containing a trivially embedded graph.

**2.4. Heegaard surfaces and  $\Gamma$ -Heegaard surfaces.** Let  $(M, T)$  be a compact connected orientable 3-manifold containing a properly embedded graph. A *Heegaard splitting* for  $(M, T)$  is a decomposition of  $M$  into two compression bodies,  $C_1$  and  $C_2$ , such that  $T_i = T \cap C_i$  is trivially embedded in  $C_i$  for  $i \in \{1, 2\}$ . The surface  $H = \partial_+ C_1 = \partial_+ C_2$  is called a *Heegaard surface* for  $(M, T)$ . We will also say that  $T$  is in *bridge position* with respect to  $H$  and that  $H$  is a *bridge splitting* of  $(M, T)$ .

Suppose now that  $\Gamma$  is a subgraph of  $T$  and  $H$  is a surface in  $(M, T)$  transverse to  $T$  so that  $H$  splits  $M$  into compression bodies  $C_1$  and  $C_2$  such that  $(C_i, T_i, \Gamma_i)$  is a  $\Gamma_i$ -compression body containing a trivially embedded graph  $T_i - \Gamma_i$  for  $i \in \{1, 2\}$ . In this case we say  $H$  is  $\Gamma$ -Heegaard surface or a  $\Gamma$ -bridge surface for  $(M, T, \Gamma)$ . If  $\Gamma = \emptyset$  then  $H$  is simply a Heegaard surface.

Suppose  $H$  is a  $\Gamma$ -Heegaard surface for  $(M, T, \Gamma)$  splitting it into triples  $(C_1, T_1, \Gamma_1)$  and  $(C_2, T_2, \Gamma_2)$ . We will say that  $H$  is  *$T$ -reducible* if there exists a sphere  $S$  disjoint from  $T$  such that  $S \cap H$  is a single curve essential in

$H - \eta(T)$ , otherwise  $H$  is  $T$ -irreducible. We will say that  $H$  is  $T$ -weakly reducible if  $H$  has  $T$ -compressing disks on opposite sides with disjoint boundaries. Otherwise  $H$  is said to be  $T$ -strongly irreducible. We will say that  $H$  is  $(T, \Gamma)$ -c-weakly reducible if  $H$  has  $(T, \Gamma)$ -c-disks on opposite sides with disjoint boundaries. Otherwise  $H$  is said to be  $(T, \Gamma)$ -c-strongly irreducible.

**2.5. Generalized stabilizations, perturbations and removable paths.** Several geometric operations can be used to produce new  $\Gamma$ -bridge surfaces from old ones. These are generalizations of stabilizations for Heegaard splittings of manifolds and usually we work with bridge surfaces that are not obtained from others via these operations. A more detailed discussion of these operations can be found in [HS2, STo, TT]. A  $\Gamma$ -bridge surface  $H$  for  $(M, T, \Gamma)$  will be called *stabilized* if there is a pair of  $T$ -compressing disks on opposite sides of  $H$  that intersect in a single point. The  $\Gamma$ -bridge surface is *meridionally stabilized* if there is a  $(T, \Gamma)$ -cut-disk and a  $T$ -compressing disk on opposite sides of  $H$  that intersect in a single point.  $H$  will be called *bimeridionally stabilized* if there are two  $(T, \Gamma)$ -cut-disks on opposite sides of  $H$  that intersect in a single point. The concept of “bimeridionally stabilized” is not used in the main theorem; it shows up only in the application.

As we are considering manifolds with boundary there are two other geometric operations that can be used to obtain a new bridge surface from an old one. Suppose  $H$  is a  $\Gamma$ -Heegaard splitting for  $(M, T, \Gamma)$  decomposing  $M$  into compression bodies  $C_1$  and  $C_2$ . Let  $F$  be a component of  $\partial_- C_1 \subset \partial M$  and let  $T'$  be a collection of vertical edges in  $F \times [-1, 0]$  so that  $T' \cap (F \times \{0\}) = T \cap F$ . Let  $H'$  be a minimal genus Heegaard surface for  $(F \times [-1, 0], T')$  which does not separate  $F \times \{-1\}$  and  $F \times \{0\}$  and which intersects each edge in  $T'$  exactly twice.  $H'$  can be formed by tubing two parallel copies of  $F$  along a vertical arc not in  $T'$ . We can form a  $\Gamma$ -Heegaard surface  $H''$  for  $M \cup (F \times [-1, 0])$  by *amalgamating*  $H$  and  $H'$ . This is simply the usual notion of amalgamation of Heegaard splittings (see [Sc]). In fact,  $H''$  is a  $\Gamma$ -Heegaard surface for  $(M \cup (F \times [-1, 0]), T \cup T')$ . Since  $(M \cup (F \times [-1, 0]), T \cup T')$  is homeomorphic to  $(M, T)$ , we may consider  $H''$  to be a  $\Gamma$ -Heegaard surface for  $(M, T, \Gamma)$ .  $H''$  is called a *boundary stabilization* of  $H$ . A similar construction can be used to obtain a new  $\Gamma$ -Heegaard splitting of  $(M, T, \Gamma)$  by tubing two parallel copies of  $F$  along a vertical arc that does lie in  $T' \subset \Gamma$ . In this case  $H''$  will be called *meridionally boundary stabilized*.

If a  $\Gamma$ -bridge surface is stabilized, boundary stabilized, meridionally stabilized, or meridionally boundary stabilized we will say that it contains a *generalized stabilization*.

A  $\Gamma$ -bridge surface is called *cancellable* if there is a pair of bridge disks  $D_i$  on opposite sides of  $H$  such that  $\emptyset \neq (\partial D_1 \cap \partial D_2) \subset (H \cap T)$ . If  $|\partial D_1 \cap$

$|\partial D_2| = 1$  we will call the bridge surface *perturbed*. Unlike the case when  $T$  is a 1-manifold, a perturbed bridge surface cannot necessarily be unperturbed by an isotopy. If neither of  $\partial D_i$  contains a vertex of  $T$ , then a new bridge surface  $H'$  for  $(M, T)$  can be found so that  $|H' \cap T| < |H \cap T|$ . If both  $\partial D_1$  and  $\partial D_2$  contain a vertex of  $T$  then a simpler bridge splitting does not necessarily exist. However, in this situation  $T$  often has a “perturbed level edge” (see below). Lemmas 9.5 and 9.6 provide more details on when perturbed bridge splittings can be unperturbed.

We say that  $T$  has a *perturbed level cycle*  $\sigma$  if there is a pair of bridge disks  $\{D_1, D_2\}$  on opposite sides of  $H$  such that  $(\partial D_1 \cap \partial D_2) \subset (T \cap H)$ , and  $\sigma \subset \partial(D_1 \cap \partial D_2)$ .  $T$  has a *perturbed level edge* if  $T$  is perturbed by disks  $D_1, D_2$  such that both  $\partial D_1$  and  $\partial D_2$  contain an interior vertex of  $T$ . The pair  $\{D_1, D_2\}$  is called the *associated cancelling pair of disks*.

Suppose that  $\zeta \subset T$  is a 1-manifold which is the union of edges in  $T$  (possibly a closed loop with zero or more vertices of  $T$ ). We say that  $\zeta$  is a *removable path* if the following hold:

- (1) Either the endpoints of  $\zeta$  lie in  $\partial M$  or  $\zeta$  is a cycle in  $T$ .
- (2)  $\zeta$  intersects  $H$  exactly twice
- (3) If  $\zeta$  is a cycle, there exists a cancelling pair of disks  $\{D_1, D_2\}$  for  $\zeta$  with  $D_j \subset C_j$ . Furthermore there exists a compressing disk  $E$  for  $H$  such that  $|E \cap T| = 1$  and if  $E \subset C_j$  then  $|\partial E \cap \partial D_{j+1}| = 1$  (indices run mod 2) and  $E$  is otherwise disjoint from a complete collection of bridge disks for  $T - H$  containing  $D_1 \cup D_2$ .
- (4) If the endpoints of  $\zeta$  lie in  $\partial M$ , there exists a bridge disk  $D$  for the bridge arc component of  $\zeta - H$  such that  $D - T$  is disjoint from a complete collection of bridge disks  $\Delta$  for  $T - H$ . Furthermore, there exists a compressing disk  $E$  for  $H$  on the opposite side of  $H$  from  $T$  such that  $|E \cap D| = 1$  and  $E$  is disjoint from  $\Delta$ .

If  $T$  has a removable path  $\zeta$ ,  $\zeta$  can be isotoped to lie in a spine for one of the compression bodies  $C_1$  or  $C_2$ .

**2.6.  $\Gamma$ -c-Heegaard surfaces and removable edges.** This section presents a technical result which sometimes allows a  $\Gamma$ -c-Heegaard surface to be converted into a Heegaard surface with removable edges.

Suppose that  $H$  is a  $\Gamma$ -Heegaard surface for  $(M, T, \Gamma)$  and that  $e \subset \Gamma$  is an edge disjoint from  $H$  with both endpoints in  $\partial M$ . Let  $E$  be a cut disk intersecting  $e$  whose boundary is in  $H$ . By isotoping  $e$  so that  $e \cap E$  moves through  $\partial E$  to the other side, we convert  $e$  into a removable path  $e'$ . Let  $T'$  be the new graph. Let  $H'$  be the new  $\Gamma$ -Heegaard surface for  $(M, T')$ . Let  $D$  be the bridge disk for the bridge arc component of  $e' - H'$ .



**Lemma 2.2.** *If  $H'$  is stabilized or meridionally stabilized then so is  $H$ . If  $H'$  is boundary-stabilized or meridionally boundary stabilized, so is  $H$  and the stabilization is along the same component of  $\partial M$ . If  $H'$  is perturbed then so is  $H$ . If  $T'$  contains a removable path other than  $e'$  then either that path is removable in  $T$  or  $H$  is meridionally stabilized.*

*Proof. Case 1:* Suppose that  $H'$  is stabilized or meridionally stabilized by disks  $D_1$  and  $D_2$  on opposite sides of  $H'$  which intersect once. Out of all such pairs of stabilizing or meridionally stabilizing pairs, choose  $D_1$  and  $D_2$  so that  $|(D_1 \cup D_2) \cap (D \cup E)|$  is minimal. The next claim shows that  $(D_1, D_2)$  also stabilizes or meridionally stabilizes  $H$ .

**Claim:**  $|(D_1 \cup D_2) \cap e| \leq 1$ .

Since  $(D_1, D_2)$  (meridionally) stabilize  $H'$ ,  $|(D_1 \cup D_2) \cap e'| \leq 1$ . Without loss of generality, choose the labelling so that  $D_1$  is on the same side of  $H$  as  $D$ . Then  $D_1 \cap e = \emptyset$ , and so  $(D_1 \cup D_2) \cap e = D_2 \cap e$ . If  $D_2$  is disjoint from and not parallel to  $E$ , then clearly  $|e \cap D_2| \leq 1$ . If  $D_2$  is parallel to  $E$ , then  $|D_2 \cap e| = |E \cap e| = 1$ . If  $D_2$  is not disjoint from  $E$ , then by the minimality of  $|(D_1 \cup D_2) \cap (E \cup D)|$  it follows that  $D_2 \cap E$  is a collection of arcs. After possibly a small isotopy, we may assume that  $D \cap E$  is disjoint from  $D_2 \cap E$ . Then using  $D$  to isotope  $e'$  back to  $e$  guarantees that  $e$  is disjoint from  $D_2$ .  $\square$ (Claim)

**Case 2:** Suppose that  $H'$  is boundary stabilized or meridionally boundary stabilized. In this case there exists a  $(T, \Gamma)$ -c-disk  $D'$  for  $H$  such that compressing  $H'$  along  $D'$  produces surfaces  $H_1$  and  $H_2$  and the surface  $H_2$  bounds a product region with  $\partial M$  containing only vertical arcs of  $T$  while the surface  $H_1$  is a  $\Gamma$ -Heegaard surface for  $(M, T, \Gamma)$ .

If  $D'$  is disjoint from or parallel to  $E$  then it is clear that  $H$  is boundary stabilized or meridionally boundary stabilized. Suppose, therefore that  $D'$  intersects  $E$ . We may assume that  $D'$  was chosen so as to minimize  $D' \cap E$ . This implies that  $D' \cap E$  is a non-empty collection of arcs. Then, as in Case 1, isotoping  $e'$  back to  $e$  shows that  $e \cap D' = \emptyset$ . Hence,  $H$  is boundary stabilized or meridionally boundary stabilized and the stabilization is along the same component as that of  $H'$ .

**Case 3:** Suppose that  $(D_1, D_2)$  are a perturbing pair of disks for  $H'$  such that  $D_1$  is on the same side of  $H$  as  $D$ . Notice that  $\partial D_1 \cup \partial D_2$  is disjoint from  $e'$  since two components of  $e' - H'$  are vertical arcs in the compression body containing them. Thus, unless  $e \cap D_2 \neq \emptyset$ ,  $(D_1, D_2)$  is a perturbing pair for  $H$ . If  $D_2 \cap e \neq \emptyset$ , then  $D_2$  intersects the neighborhood of  $E$  used to push  $e$  to  $e'$ . An argument similar to that of Cases 1 and 2 shows that  $\partial D$  can be assumed to be disjoint from  $\partial D_2$ , and so  $e$  is disjoint from  $D_2$ , as desired.

**Case 4:** Suppose that  $T'$  contains a removable path  $\zeta \neq e$ . An argument similar to the previous cases shows that  $\zeta$  is a removable path in  $T$ , unless  $\zeta$  is not a cycle and the compressing disk  $E$  from condition (4) of the definition of removable path is equal to the present disk  $E$ . Suppose, therefore, that this is the case. By the definition of removable path, there is a bridge disk  $D'$  for  $e' - H'$  which is disjoint from  $E$ . A small isotopy of  $D \cup D'$  creates a compressing disk  $E'$  for  $H' - T'$  intersected once by  $E$ . The pair  $(E, E')$  therefore, shows that  $H'$  is stabilized. Since  $\partial E'$  is non-separating on  $H'$  and therefore on  $H$ ,  $E'$  is a compressing disk for  $H$  disjoint from  $T$ . The boundary of  $E'$  intersects  $E$  exactly once and  $E \cap T = E \cap e$  and so  $H$  is meridionally stabilized.  $\square$

**2.7. Multiple bridge splittings.** To prove our main theorem we will extend to graphs the definition of multiple bridge splittings for the pair (3-manifold, 1-manifold) introduced by Hayashi and Shimokawa in [HS3] and generalized by Tomova in [T].

**Definition 2.3.** Suppose  $M$  is a 3-manifold containing a properly embedded graph  $T$  and let  $\Gamma$  be a subgraph of  $T$ . A disjoint union of surfaces  $\mathcal{H}$  is a multiple  $\Gamma$ -Heegaard splitting for  $(M, T, \Gamma)$  if:

- the closure of each component of  $(M, T) - \mathcal{H}$  is a  $\Gamma$ -compression body  $C_i$  containing a trivially embedded graph  $T_i = (T - \Gamma) \cap C_i$ ,
  - for each  $i$ ,  $\partial_+ C_i$  is attached to some  $\partial_+ C_j$  and  $\partial_- C_i$  is either contained in  $\partial M$  or is attached to some  $\partial_- C_k$ .
- Let  $\mathcal{H}^+ = \cup \partial_+ C_i$  and  $\mathcal{H}^- = \cup \partial_- C_i$

### 3. PROPERTIES OF COMPRESSION BODIES CONTAINING PROPERLY EMBEDDED GRAPHS

In this section we will generalize many of the well known results for compression bodies to the case when the compression body contains a graph embedded in a specific way.

**Lemma 3.1.** Suppose  $C$  is a  $\Gamma$ -compression body containing a trivially embedded graph  $T - \Gamma$ . Compressing or cut-compressing  $C$  results in a union of  $\Gamma$ -compression bodies each containing a trivially embedded graph.

*Proof.* Given any  $(T, \Gamma)$ -c-disk  $D^c$  for  $(C, T, \Gamma)$  we can always find a collection of pairwise disjoint  $(T, \Gamma)$ -c-disks  $\mathcal{D}^c$  containing  $D^c$  so that  $(C, T, \Gamma)$  c-compressed along  $\mathcal{D}^c$  is a collection 3-balls and components homeomorphic to  $G \times I$  where  $G$  is a component of  $\partial_- C$ . Both types of components may contain trivially embedded graphs. The result follows.  $\square$

The next lemma will imply that the negative boundary of a  $\Gamma$ -compression body is incompressible in the complement of  $T$ .



**Lemma 3.2.** *Suppose  $(C, T, \Gamma)$  is a  $\Gamma$ -compression body containing a trivially embedded graph  $T - \Gamma$ . If  $F$  is a  $(T, \Gamma)$ -c-incompressible,  $T$ - $\partial$ -incompressible surface in  $C$  transverse to  $T$ , then  $F$  is a collection of the following kinds of components:*

- *spheres that bound balls in  $C$ : each ball may intersect  $T$  in at most one component and this component can include at most one vertex of  $T$ ,*
- *disks intersecting  $T$  in 0 or 1 points,*
- *vertical annuli disjoint from  $T$ ,*
- *closed surfaces parallel to components of  $\partial_- C$  so that the region of parallelism intersects the graph in vertical arcs.*

*Proof.* Let  $\mathcal{D}^c$  be the collection of  $(T, \Gamma)$ -c-disks so that  $C$  compressed along  $\mathcal{D}^c$  is a union of pairs  $(C_i, T_i)$  where  $C_i$  is a compression body and  $T_i$  is a graph trivially embedded in  $C_i$ . As  $F$  is  $(T, \Gamma)$ -c-incompressible and  $T$ - $\partial$ -incompressible, it can be isotoped to be disjoint from  $\mathcal{D}^c$ . Without loss of generality suppose  $F \subset C_1$ .

Suppose  $\tau$  is a component of  $T_1$  which is a pod or a vertical pod. Let  $D$  be a pod disk for  $\tau$  chosen so that  $|F \cap D|$  is minimal. Because  $F$  is incompressible and  $\partial$ -incompressible  $F$  cannot intersect  $D$  in arcs or simple closed curves disjoint from  $\tau$ . In fact if  $F$  intersects  $D$  at all,  $F$  must be either a once punctured disk intersecting the pod disk in an arc  $\alpha$  so that  $|\alpha \cap l| = 1$  where  $l$  is a pod leg of  $\tau$ , a twice punctured sphere that intersects the pod disk in a simple closed curve  $\beta$  so that  $\beta \cap l = 2$ , or a sphere that bounds a ball containing the interior vertex of  $\tau$ . In the last case  $F \cap D$  is a single simple closed curve that is the boundary of a neighborhood of the vertex.

We conclude that either  $F$  is one of the first three types of components or it is disjoint from all pod-disks. In the latter case  $F$  is contained in one of the components obtained from  $C_1$  by removing all pod-disks. Therefore  $F$  is either contained in a ball disjoint from  $T$  or in the compression body  $(C', T')$  where  $C'$  is isotopic to  $C_1$  and  $T'$  is a trivially embedded tangle in  $C'$ . In either case the desired result follows by [HS2, Lemma 2.4].  $\square$

**Corollary 3.3.** *If  $(C, T, \Gamma)$  is a  $\Gamma$ -compression body containing a trivially embedded graph  $T - \Gamma$ , then  $\partial_- C$  is  $T$ -incompressible.*

**Lemma 3.4.** *Suppose  $(C, T, \Gamma)$  is a  $\Gamma$ -compression body containing a trivially embedded graph  $T - \Gamma$ . If  $F$  is a connected  $T$ -incompressible surface such that  $F \cap \partial C = \emptyset$ , then  $F$  bounds a compression body  $C_F$  in  $C$  so that  $\partial_- C_F \subset \partial_- C$  and  $F$  is  $T$ -parallel to  $\partial_- C_F$ .*

*Proof.* If  $F$  is a  $(T, \Gamma)$ -c-incompressible surface, the result follows from Lemma 3.2 so suppose this is not the case. Maximally  $(T, \Gamma)$ -cut-compress

$F$  and note that  $(T, \Gamma)$ -c-compressing never creates  $T$ -compressing disks for the surface. Therefore, by Lemma 3.2 the resulting collection of  $(T, \Gamma)$ -c-incompressible surfaces  $\mathcal{F}$  consists of two kinds of components: spheres that bound balls in  $C$  so that each ball intersects  $T$  in at most one component and this component contains at most one vertex of  $T$ , and closed surfaces parallel to components of  $\partial_- C$  so that the regions of parallelism intersect the graph in vertical arcs. Furthermore note that each component of  $\mathcal{F}$  intersects  $\Gamma$  in at least one point so there are no sphere components disjoint from  $T$ .

Suppose  $\mathcal{F}$  contains a sphere component  $S'$  which bounds a ball  $B$  that intersects  $T$  in a single edge. As cut-disks have boundaries that are essential curves in  $F - T$ , such a sphere can only be the result of cut-compressing a torus which is the boundary of a regular neighborhood of a closed component of  $T$  with no vertices. As this torus is disjoint from  $T$ , it is in fact the original surface  $F$  and the result follows. Therefore from now on we may assume that any sphere components of  $\mathcal{F}$  bound balls that contain a vertex of  $T$ .

The surface  $F$  can be recovered from  $\mathcal{F}$  by adding to  $\mathcal{F}$  a collection of tunnels  $\Lambda$  that run along edges of  $\Gamma$  and are dual to the cut-disks. Let  $F'$  be an outermost component that is  $T$ -parallel to a component of  $\partial_- C$ , i.e.,  $F'$  is not contained in the region of parallelism of any other component. Let  $P'$  be the region of parallelism between  $F'$  and some component of  $\partial_- C$ . By Lemma 3.2 there are no vertices of  $T$  in  $P'$  and therefore none of the sphere components of  $\mathcal{F}$  are contained in  $P'$ . Suppose  $F''$  is a component of  $\mathcal{F}$  contained in  $P'$  and adjacent to  $F'$ . As  $F$  is connected there must be  $\lambda \in \Lambda$  connecting  $F'$  and  $F''$ . Let  $\alpha$  be any essential curve in  $F' - \eta(T)$  with both endpoints at  $\lambda \cap F'$ . As  $F'$  and  $F''$  are  $T$ -parallel to the same component of  $\partial_- C$ , they are  $T$ -parallel to each other. This parallelism gives a rectangle  $D$  with two opposite sides running along  $\lambda$ , one side in  $\alpha$  and the last side is the image of  $\alpha$  in  $F''$  under the parallelism. As  $D$  does not intersect  $T$  and therefore doesn't intersect any of the tubes in  $\Lambda$ , it is a  $T$ -compressing disk for  $F$ , a contradiction. Therefore all regions of  $T$ -parallelism between components of  $\mathcal{F}$  and components of  $\partial_- C$  are disjoint from all other components of  $\mathcal{F}$ .

Consider now a sphere component  $S$  of  $\mathcal{F}$  bounding a ball  $B$ . If  $B$  contains any other component  $S'$  of  $\mathcal{F}$ , this component must be a sphere  $T$ -parallel to  $S$ . Note that as  $B$  contains a vertex,  $S$  and  $S'$  have at least three punctures each. We can now repeat the argument above to show that  $F$  is  $T$ -compressible. Therefore we conclude that each component  $F'$  of  $\mathcal{F}$  bounds a compression body  $C_{F'}$  which is either a ball intersecting  $T$  in a single vertex or is homeomorphic to  $F' \times I$  and intersects  $T$  in vertical arcs. All of these compression bodies are disjoint and are also disjoint from all

tubes  $\Lambda$ . This also implies that each tube in  $\Lambda$  is parallel to a different edge of  $\Gamma$ . The union of the compression bodies and the parallelisms between each tube in  $\Lambda$  and some edge of  $\Gamma$  gives the desired compression body  $C_F$ .  $\square$

Finally we recall the classification of Heegaard splittings of pairs  $(C, T)$  where  $\partial_+ C$  is  $T$ -parallel to  $\partial_- C$ , [TT]. This result is key to proving our main theorem.

**Theorem 3.5.** [TT, Theorem 3.1] *Let  $M$  be a compression body and  $T$  be a properly embedded graph so that  $\partial_+ M$  is  $T$ -parallel to  $\partial_- M$ . Let  $H$  be a Heegaard surface for  $(M, T)$ . Assume that  $T$  contains at least one edge. Then one of the following occurs:*

- (1)  $H$  is stabilized;
- (2)  $H$  is boundary stabilized;
- (3)  $H$  is perturbed;
- (4)  $T$  has a removable path disjoint from  $\partial_+ M$ ;
- (5)  $M$  is a 3-ball,  $T$  is a tree with a single interior vertex (possibly of valence 2), and  $H - \mathring{\eta}(T)$  is parallel to  $\partial M - \mathring{\eta}(T)$  in  $M - \mathring{\eta}(T)$ ;
- (6)  $M = \partial_- M \times I$ ,  $H$  is isotopic in  $M - \mathring{\eta}(T)$  to  $\partial_+ M - \mathring{\eta}(T)$ .

For our purposes we need to strengthen the second conclusion of the above theorem:

**Theorem 3.6.** *Let  $M$  be a compression body and  $T$  be a properly embedded graph so that  $\partial_+ M$  is  $T$ -parallel to  $\partial_- M$ . Let  $H$  be a Heegaard surface for  $(M, T)$ . Assume that  $T$  contains at least one edge. Then one of the following occurs:*

- (1)  $H$  is stabilized;
- (2)  $H$  is boundary stabilized along  $\partial_- M$ ;
- (3)  $H$  is perturbed;
- (4)  $T$  has a removable path disjoint from  $\partial_+ M$ ;
- (5)  $M$  is a 3-ball,  $T$  is a tree with a single interior vertex (possibly of valence 2), and  $H - \mathring{\eta}(T)$  is parallel to  $\partial M - \mathring{\eta}(T)$  in  $M - \mathring{\eta}(T)$ ;
- (6)  $M = \partial_- M \times I$ ,  $H$  is isotopic in  $M - \mathring{\eta}(T)$  to  $\partial_+ M - \mathring{\eta}(T)$ .

*Proof.* Suppose  $H$  is boundary stabilized along  $\partial_+ M$ . Then  $H$  is obtained by amalgamating a minimal genus Heegaard surface for  $\partial_+ M \times [-1, 0]$  which does not separate  $\partial_+ M \times \{-1\}$  and  $\partial_+ M \times \{0\}$  and which intersects each edge in  $T \cap (\partial_+ M \times [-1, 0])$  exactly twice, together with a Heegaard surface  $\tilde{H}$  for  $M$ . Without loss of generality we will assume that  $H$  is obtained from  $\tilde{H}$  after a single boundary stabilization along  $\partial_+ M$ .

By Theorem 3.5,  $\tilde{H}$  satisfies one of six possible conclusions. If  $\tilde{H}$  is stabilized, perturbed or if  $T$  has a removable path disjoint from  $\partial_+ M$  then the

same is true for  $H$  as boundary stabilizations preserve all of these properties. If  $\tilde{H}$  is boundary stabilized, the stabilization must be along  $\partial_- M$ , and so the same is true for  $H$ .

Suppose that  $M$  is a 3-ball,  $T$  is a tree with a single interior vertex, and  $H - \mathring{\eta}(T)$  is parallel to  $\partial M - \mathring{\eta}(T)$  in  $M - \mathring{\eta}(T)$ . Let  $A$  and  $B$  be the components of  $M - \tilde{H}$  so that  $B$  is a ball and let  $\kappa$  be one of the edges of  $T \cap M$ . Then  $H$  can be recovered from  $\tilde{H}$  by tubing  $H$  to the boundary of a collar of  $\partial M$  along a vertical tube  $\tau$  in  $A$ . We can choose  $\tau$  to be arbitrarily close to  $\kappa \cap A$ ; in particular, we may assume that the disk of parallelism between  $\tau$  and  $\kappa$  intersects some bridge disk that contains  $\kappa \cap B$  in its boundary only in the point  $\kappa \cap \tilde{H}$ . We conclude that  $H$  is perturbed.

Suppose then that  $M = \partial_- M \times I$  and  $H$  is isotopic in  $M - \mathring{\eta}(T)$  to  $\partial_+ M - \mathring{\eta}(T)$ . Let  $A$  and  $B$  be the components of  $M - \tilde{H}$  so that  $A$  contains  $\partial_+ M$ . The argument in this case is identical to the one above as long as there is at least one bridge disk in  $B$ . If  $T \cap B$  is a product, then  $T \cap M$  is a collection of vertical arcs and thus  $H$  can be obtained from the Heegaard surface  $\tilde{H}$  by stabilizing along  $\partial_- M$ .  $\square$

#### 4. HAKEN'S LEMMA

Suppose  $(M, T)$  is a 3-manifold containing a properly embedded graph. Let  $H$  be a  $\Gamma$ -bridge surface for  $(M, T, \Gamma)$  and suppose  $D$  is a  $T$ -compressing disk for some component  $G$  of  $\partial M$ . It is a classic result of Haken [H] that in the case  $T = \emptyset$ , there is a compressing disk  $D'$  for  $G$  so that  $D'$  intersects  $H$  in a unique essential curve. This result was extended to the case where  $T$  is a tangle and  $\Gamma = \emptyset$  in [HS3] and to the case where  $T$  is a tangle and  $\Gamma = T$  in [T]. In this paper we will need a further extension of the result to include properly embedded graphs. It turns out that the proof of [T, Theorem 6.2] carries over to this situation without any modifications so we will not include it here.

**Theorem 4.1.** *Suppose  $M$  is a compact orientable manifold,  $T$  is a properly embedded graph in  $M$  and  $\Gamma$  is a subgraph of  $T$ . Assume that  $M - T$  is irreducible and that no sphere in  $M$  intersects  $T$  transversally exactly once. Suppose  $H$  is a  $\Gamma$ -Heegaard splitting for  $(M, T, \Gamma)$ . If  $D$  is a  $T$ -compressing disk for some component of  $\partial M$  then there exists such a disk  $D'$  so that  $D'$  intersects  $H - \mathring{\eta}(T)$  in a unique essential simple closed curve.*

#### 5. MULTIPLE $\Gamma$ -BRIDGE SPLITTINGS OF $(M, T, \Gamma)$

##### 5.1. Complexity.

**Definition 5.1.** *Let  $X$  be a set with an order  $\leq$ . Let  $Y$  and  $Z$  be two finite multisets of elements of  $X$ . Write  $Y = (y_1, y_2, \dots, y_n)$  and  $Z = (z_1, z_2, \dots, z_m)$*

so that for all  $i$ ,  $y_i \geq y_{i+1}$  and  $z_i \geq z_{i+1}$ . We say that  $Y < Z$  if and only if one of the following occurs:

- There exists  $j \leq \min(n, m)$  so that for all  $i < j$ ,  $y_i = z_i$  and  $y_j < z_j$ .
- $n < m$  and for all  $i \leq n$ ,  $y_i = z_i$ .

The set of all  $\Gamma$ -multiple bridge splittings for  $(M, T, \Gamma)$  can be ordered using the definition of complexity introduced in [T].

**Definition 5.2.** Let  $S$  be a closed connected surface embedded in  $M$  transverse to a properly embedded graph  $T \subset M$ . The complexity of  $S$  is the ordered pair  $c(S) = (2 - \chi(S - \eta(T)), \text{genus}(S))$ . If  $S$  is not connected,  $c(S)$  is the multi-set of ordered pairs corresponding to each of the components of  $S$ .

If  $\mathcal{H}$  is a  $\Gamma$ -multiple bridge splitting for  $(M, T, \Gamma)$ , let the complexity of  $\mathcal{H}$ ,  $c(\mathcal{H})$  be the multiset  $\{c(S) | S \in \mathcal{H}^+\}$ . If  $\mathcal{H}$  and  $\mathcal{H}'$  are two multiple  $\Gamma$ -Heegaard splittings for  $(M, T, \Gamma)$ , their complexities will be compared as in Definition 5.1.

**Lemma 5.3.** [T, Lemma 4.4] Suppose  $S$  is meridional surface in  $(M, T)$  of non-positive euler characteristic. If  $S'$  is a component of the surface obtained from  $S$  by compressing along a  $c$ -disk, then  $c(S) > c(S')$ .

The next lemma is immediate from the definition of complexity.

**Lemma 5.4.** Suppose that  $\mathcal{H}$  is a multiple  $\Gamma$ -Heegaard splitting of  $(M, T, \Gamma)$  and suppose that  $\mathcal{J}$  is a multiple  $\Gamma$ -Heegaard splitting of  $(M, T, \Gamma)$  such that  $\mathcal{J}^+$  is a proper subset of  $\mathcal{H}^+$ . Then the complexity of  $\mathcal{J}$  is strictly less than the complexity of  $\mathcal{H}$ .

## 6. UNTELESCOPING AND CONSOLIDATION

We will be interested in obtaining a multiple  $\Gamma$ -Heegaard splitting of  $(M, T, \Gamma)$  with the property that every thin surface is incompressible. The following lemma will be useful. Its proof is straightforward and similar to the proof of [T, Corollary 6.3].

**Lemma 6.1.** Suppose that  $\mathcal{H}$  is a multiple  $\Gamma$ -Heegaard splitting of  $(M, T, \Gamma)$ . If some component of  $\mathcal{H}^-$  is compressible in  $M$  then one of the following occurs:

- Some component of  $\mathcal{H}^-$  is parallel to a component of  $\mathcal{H}^+$ ; or
- Some component of  $\mathcal{H}^+$  induces a  $(T, \Gamma)$ - $c$ -weakly reducible  $\Gamma$ -Heegaard splitting of the component of  $M - \mathcal{H}^-$  containing it.

The next two subsections present two operations; each corresponds to one of the two possible conclusions in Lemma 6.1.

**6.1. Untelesoping.** In [ST1] Scharlemann and Thompson discussed the operation of *untelesoping* a weakly reducible Heegaard surface for a manifold to obtain a multiple Heegaard surface (i.e., generalized Heegaard splitting). The concept was generalized to weakly reducible bridge surfaces for a manifold containing a properly embedded tangle in [HS3] and then to  $(\tau, \tau)$ -c-weakly reducible  $\tau$ -bridge surfaces for a manifold containing a properly embedded tangle  $\tau$  in [T]. In the following definition we extend the construction to  $(T, \Gamma)$ -c-weakly reducible  $\Gamma$ -bridge surfaces for  $(M, T, \Gamma)$  where  $T$  is a properly embedded graph.

Let  $H$  be a  $(T, \Gamma)$ -c-weakly reducible  $\Gamma$ -bridge splitting of  $(M, T, \Gamma)$  and let  $\mathcal{D}_1$  and  $\mathcal{D}_2$  be collections of pairwise disjoint  $(T, \Gamma)$ -c-disks above and below  $H$  such that  $\mathcal{D}_1 \cap \mathcal{D}_2 = \emptyset$ . Then we can obtain a multiple  $\Gamma$ -bridge splitting for  $(M, T)$  with one thin surface obtained from  $H$  by  $(T, \Gamma)$ -c-compressing it along  $\mathcal{D}_1 \cup \mathcal{D}_2$  and two thick surfaces, obtained by  $(T, \Gamma)$ -c-compressing  $H$  along  $\mathcal{D}_1$  and  $\mathcal{D}_2$  respectively. Although this operation is similar to the one described in [T] we include the details here with the modifications required.

Let  $(A, A \cap T)$  and  $(B, B \cap T)$  be the two  $\Gamma$ -compression bodies into which  $H$  decomposes  $(M, T, \Gamma)$  and let  $\mathcal{D}_A \subset A$  and  $\mathcal{D}_B \subset B$  be collections of pairwise disjoint  $(T, \Gamma)$ -c-disks such that  $\mathcal{D}_A \cap \mathcal{D}_B = \emptyset$ . Let  $A' = A - \dot{\eta}(\mathcal{D}_A)$  and  $B' = B - \dot{\eta}(\mathcal{D}_B)$ . Then by Lemma 3.1  $A'$  and  $B'$  are each the disjoint union of  $\Gamma$ -compression bodies containing trivial graphs  $A' \cap T$  and  $B' \cap T$  respectively.

Take small collars  $\eta(\partial_+ A')$  of  $\partial_+ A'$  and  $\eta(\partial_+ B')$  of  $\partial_+ B'$ . Let  $C^1 = cl(A' - \eta(\partial_+ A'))$ ,  $C^2 = \eta(\partial_+ A') \cup \eta(\mathcal{D}_B)$ ,  $C^3 = \eta(\partial_+ B') \cup \eta(\mathcal{D}_A)$  and  $C^4 = cl(B' - \eta(\partial_+ B'))$ . Note that  $C_1$  and  $C_4$  are  $\Gamma$ -compression bodies containing trivial graphs because they are homeomorphic to  $A'$  and  $B'$  respectively.  $C_2$  and  $C_3$  are obtained by taking surface  $\times I$  containing vertical arcs and attaching 2-handles, some of which may contain segments of  $\Gamma$  as their cores. Therefore  $C_2$  and  $C_3$  are also  $\Gamma$ -compression bodies containing trivial graphs. We conclude that we have obtained a multiple  $\Gamma$ -bridge splitting  $\mathcal{H}$  of  $(M, T, \Gamma)$  with positive surfaces  $\partial_+ C_1$  and  $\partial_+ C_2$  that can be obtained from  $H$  by  $(T, \Gamma)$ -c-compressing along  $\mathcal{D}_A$  and  $\mathcal{D}_B$  respectively and a negative surface  $\partial_- C_2 = \partial_- C_3$  obtained from  $H$  by  $(T, \Gamma)$ -c-compressing along both sets of c-disks. We say that  $\mathcal{H}$  is obtained by untelesoping  $H$  using  $(T, \Gamma)$ -c-disks. The next remark follows directly from Lemma 5.3.

**Remark 6.2.** Suppose  $\mathcal{H}'$  is a multiple  $\Gamma$ -bridge splitting of  $(M, T, \Gamma)$  obtained from another multiple  $\Gamma$ -bridge splitting  $\mathcal{H}$  of  $(M, T, \Gamma)$  via untelesoping. Then  $c(\mathcal{H}') < c(\mathcal{H})$ .

Suppose that  $H$  is a multiple  $\Gamma$ -Heegaard splitting for  $(M, T, \Gamma)$  which can be untelesoped to become a multiple  $\Gamma$ -Heegaard splitting  $\mathcal{H}$  for



$(M, T, \Gamma)$ . It is natural to ask: If  $\mathcal{H}$  contains a generalized stabilization must  $H$  contain a generalized stabilization? If  $\mathcal{H}$  contains a perturbed thick surface must  $H$  also be perturbed? If  $T$  has a path which is removable with respect to  $\mathcal{H}$ , is that path removable with respect to  $H$ ? The answer is positive in all cases.

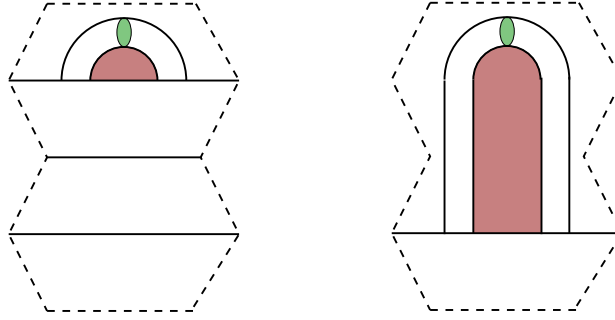
**Lemma 6.3.** *Let  $\mathcal{H}$  be a multiple  $\Gamma$ -Heegaard splitting for the triple  $(M, T, \Gamma)$  obtained by untelescoping the (multiple)  $\Gamma$ -Heegaard splitting  $H$  for  $(M, T, \Gamma)$ .*

- (1) *Suppose the  $\Gamma$ -Heegaard splitting induced by  $\mathcal{H}^+$  on some component of  $(M, T, \Gamma) - \mathcal{H}^-$  contains a generalized stabilization. Furthermore suppose that if the generalized stabilization is a (meridional) boundary stabilization, then it is along a component of  $\partial M$ . Then  $H$  contains a generalized stabilization of the same type and if the generalized stabilization is a (meridional) boundary stabilization then it must also be along a component of  $\partial M$ .*
- (2) *If the  $\Gamma$ -Heegaard splitting induced by  $\mathcal{H}^+$  on some component of  $(M, T, \Gamma) - \mathcal{H}^-$  is perturbed, then so is  $H$ .*
- (3) *If the  $\Gamma$ -Heegaard splitting induced by  $\mathcal{H}^+$  on some component of  $(M, T, \Gamma) - \mathcal{H}^-$  contains a removable path so that the path is either a cycle or both of its endpoints are contained in  $\partial M$ , then so does  $H$ .*

*Proof.* Without the loss of much generality, we may assume that  $\mathcal{H}$  is obtained from  $H$  by a single untelescoping operation. Suppose that  $H$  decomposes  $(M, T)$  into compression bodies  $A$  and  $B$ . Let  $N_i$  for  $i = 1, 2$  be the closure of the components of  $M - \mathcal{H}^-$  on either side of  $\mathcal{H}^-$ . Let  $H_i$  be the  $\Gamma$ -Heegaard surface of  $N_i$  induced by  $\mathcal{H}^+$ . Assume that  $H_1$  contains a generalized stabilization, or is perturbed, or that  $T \cap N_1$  contains a removable path satisfying the hypotheses of (3).

Consider a collar  $\mathcal{H}^- \times [-1, 1]$  of  $\mathcal{H}^-$  where  $\mathcal{H}^- = \mathcal{H}^- \times \{0\}$ . Recall that  $H_1$  can be obtained from  $\mathcal{H}^- \times [0, 1]$  by attaching handles (possibly with cores running along the knot) to  $\mathcal{H}^- \times \{1\}$ . Similarly,  $H_2$  can be obtained from  $\mathcal{H}^- \times [-1, 0]$  by attaching handles to  $\mathcal{H}^- \times \{-1\}$ . The Heegaard surface  $H$  can be obtained by extending the handles of  $H_1$  through  $\mathcal{H}^- \times [-1, 1]$  and attaching them to  $H_2$ .

**Case 1A:**  $H_1$  is stabilized or meridionally stabilized. Let  $D$  and  $E$  be the  $(T, \Gamma)$ -c-disks with boundary on  $H_1$  defining the (meridional) stabilization. We may assume that the handles attached to  $\mathcal{H}^- \times \{1\}$  include one that has  $\partial D$  as a core. The intersection of  $\partial E$  with  $\mathcal{H}^- \times \{1\}$  then consists of a single arc. The possibly once punctured disk  $E' = E \cup (\partial E \times [-1, 1])$  is then a c-disk with boundary on  $H$ . The c-disks  $E'$  and  $D$  intersect exactly once and define a (meridional) stabilization of  $H$ . See Figure 1.

FIGURE 1. If  $H_1$  is stabilized, so is  $H$ .

**Case 1B:**  $H_1$  is boundary stabilized or meridionally boundary stabilized. Let  $C_1^+$  and  $C_2^+$  be the two  $\Gamma$ -compression bodies into which  $H_1$  divides  $N_1$ . Without loss of generality in this case  $C_1^+$  contains some component  $G \subset \partial N$  and a c-disk  $D$  so that removing  $\eta(D)$  from  $C_1^+$  decomposes it into a compression body  $C_1'^+$  and a component  $R$  homeomorphic to  $G \times I$  and adding  $R$  to  $C_2^+$  results in a compression body  $C_2'^+$ . Let  $\tilde{H}_1 = \partial_+ C_1'^+ = \partial_+ C_2'^+$ . Amalgamating the multiple bridge splitting with thick surfaces  $\tilde{H}_1$  and  $H_1$  gives a bridge splitting for  $N$  with bridge surface  $\tilde{H}$  which can be obtained from  $H$  by c-compressing along the disk  $D$ . Therefore  $H$  is boundary stabilized or meridionally boundary stabilized.

**Case 2:**  $H_1$  is perturbed. Let  $D$  and  $E$  be the two bridge disks for  $H_1$  that intersect in one or two points and these points lie in  $T$ . Both disks are completely disjoint from  $\mathcal{H}^- \times \{1\}$  so in particular extending the 1-handles from  $H_1$  across  $\mathcal{H}^- \times [-1, 1]$  has no effect on these disks. Therefore  $H$  is also perturbed.

**Case 3:** Suppose  $\zeta$  is a removable path in  $N_1$  and suppose first that  $\zeta$  is a cycle. Let  $C_1$  and  $C_2$  be the two  $\Gamma$ -compression bodies  $\text{cl}(N_1 - H_1)$ . As  $\zeta$  is a removable cycle there exists a cancelling pair of disks  $\{D_1, D_2\}$  for  $\zeta$  with  $D_j \subset C_j$ . These disks are both completely disjoint from  $\mathcal{H}^- \times \{1\}$  so in particular extending the 1-handles from  $H_1$  across  $\mathcal{H}^- \times [-1, 1]$  has no effect on these disks. As  $\zeta$  is a removable cycle there exists a compressing disk  $E$  in  $C_1$ , say, so that  $|E \cap T| = 1$  and  $|\partial E \cap \partial D_2| = 1$  and  $E$  is otherwise disjoint from a complete collection of bridge disks for  $(T \cap N_1) - H_1$  containing  $D_1 \cup D_2$ . We may assume that the handles attached to  $\mathcal{H}^- \times \{1\}$  include one that has  $\partial E$  as a core. Therefore  $E$  satisfies all the desired properties as a compressing disk for  $H$ .

Suppose then that  $\zeta$  is a path. Again we may assume that the handles attached to  $\mathcal{H}^- \times \{1\}$  include one that has  $\partial E$  as a core. The bridge disk for the component of  $\zeta - H_1$  is also a bridge disk for  $\zeta - H$  satisfying all of the required properties.  $\square$

**6.2. Consolidation.** Suppose that  $\mathcal{H}$  is a multiple  $\Gamma$ -Heegaard surface for  $(M, T, \Gamma)$  and that there is a component  $F \subset \mathcal{H}^-$  which is parallel to a component  $H$  of  $\mathcal{H}^+$ . That is,  $F$  and  $H$  cobound a submanifold  $C$  homeomorphic to  $F \times I$  such that  $C \cap T$  consists of vertical edges. Recall that  $\partial_- C = F$  and  $\partial_+ C = H$ . Let  $\mathcal{H}' = \mathcal{H} - (F \cup H)$ . If the compression body of  $M - \mathcal{H}$  adjacent to  $F$  but not to  $H$  does not contain vertical pods with handles adjacent to  $F$ , we say that  $\mathcal{H}'$  is obtained from  $\mathcal{H}$  by *consolidation*.

**Lemma 6.4.** *Suppose that  $\mathcal{H}'$  is obtained from  $\mathcal{H}$  by consolidation. Then  $\mathcal{H}'$  is a multiple  $\Gamma$ -Heegaard splitting of  $(M, T, \Gamma)$ . Furthermore, the complexity of  $\mathcal{H}'$  is strictly less than the complexity of  $\mathcal{H}$ .*

*Proof.* Let  $C_+$  be the compression body adjacent to  $\partial_+ C$  and let  $C_-$  be the compression body adjacent to  $\partial_- C$ . We can view  $C_-$  as obtained by adding 1-handles to  $\partial_- C$  where some of these 1-handles may have segments of  $\Gamma$  as their cores. These 1-handles can be extended through the product structure of  $C$  to be considered as added to  $\partial_+ C = \partial_+ C_+$  and then they can be further extended to be added to  $\partial_- C_+$ . Thus, the union of  $C$ ,  $C_-$ , and  $C_+$  is a compression body. Since  $T \cap C$  is a collection of vertical edges and since  $T \cap C_+$  contains no vertical pods with handles adjacent to  $F$ ,  $T \cap (C_- \cup C \cup C_+)$  is trivially embedded in  $C_- \cup C \cup C_+$ . By Lemma 5.4, the complexity of  $\mathcal{H}'$  is strictly less than the complexity of  $\mathcal{H}$ .  $\square$

**6.3. Combining untelescoping and consolidation.** We will usually use consolidation in conjunction with untelescoping, so the next lemma is important.

**Lemma 6.5.** *Suppose that  $\mathcal{H}$  is a multiple  $\Gamma$ -Heegaard surface obtained by untelescoping and consolidating a  $\Gamma$ -Heegaard surface  $H$  for  $(M, T, \Gamma)$ . If  $e$  is an edge in  $M - \mathcal{H}$  which is a pod handle then  $e$  is a pod handle in  $M - H$ . In particular,  $e$  is adjacent to  $\partial M$ .*

*Proof.* First note that consolidation never introduces pod-handles that are adjacent to a thin surface.

Let  $A$  and  $B$  be the compression bodies which are the closures of the components of  $M - H$ . Let  $D$  be a  $(T, \Gamma)$ -c-disk in  $A$ . Let  $A'$  be obtained by reducing  $A$  using  $D$  and let  $H'$  be a copy of  $\partial_+ A'$  pushed slightly into  $A'$ . The surface  $H'$  is a  $\Gamma$ -Heegaard surface for  $A'$ . Let  $A'_1$  be the compression body of  $A' - H'$  adjacent to  $\partial_+ A'$ . Each component of  $T \cap A'$  is a vertical edge, since, in creating  $H'$ , we did not push  $\partial_+ A'$  past any vertices of  $T \cap A'$ .

Suppose then that  $E$  is a  $(T, \Gamma)$ -c-disk in  $B$ . Let  $A''$  be obtained from  $A'$  by attaching a regular neighborhood of  $E$  to  $\partial_+ A'$ . The surface  $H'$  is still a  $\Gamma$ -Heegaard surface for  $A''$ . If  $E$  was a  $T$ -compressing disk then

$T \cap A'' = T \cap A'$ . If  $E$  was a  $(T, \Gamma)$ -cut disk then  $T \cap A''$  contains one more edge than  $T \cap A'$ . The new edge is disjoint from  $H$  and has both endpoints on  $\partial A''$ . It is, therefore, not a pod handle in  $A'' - H'$ . The lemma follows immediately from these observations.  $\square$

Suppose that  $\mathcal{J}$  is a multiple  $\Gamma$ -Heegaard splitting of  $(M, T, \Gamma)$  obtained by untelescoping a  $\Gamma$ -Heegaard surface  $H$  for  $(M, T, \Gamma)$ . If a component of  $\mathcal{J}^-$  is parallel to a component of  $\mathcal{J}^+$  then we may consolidate  $\mathcal{J}$  to obtain a multiple  $\Gamma$ -Heegaard splitting  $\mathcal{K}$  of  $(M, T, \Gamma)$  such that no component of  $\mathcal{K}^-$  is parallel to a component of  $\mathcal{K}^+$ . The proof of the next lemma is similar to that of Lemma 6.3 and so we omit it.

**Lemma 6.6.** *Use the above notation.*

- Suppose the  $\Gamma$ -Heegaard splitting induced by  $\mathcal{K}^+$  on some component of  $(M, T, \Gamma) - \mathcal{K}^-$  contains a generalized stabilization. Furthermore, suppose that if the generalized stabilization is a (meridional) boundary stabilization, then it is along a component of  $\partial M$ . Then  $H$  contains a generalized stabilization of the same type and if the generalized stabilization is a (meridional) boundary stabilization then it must also be along a component of  $\partial M$ .
- If the  $\Gamma$ -Heegaard splitting induced by  $\mathcal{K}^+$  on some component of  $(M, T, \Gamma) - \mathcal{K}^-$  is perturbed then so is  $H$ .
- If the  $\Gamma$ -Heegaard splitting induced by  $\mathcal{K}^+$  on some component of  $(M, T, \Gamma) - \mathcal{K}^-$  contains a removable path which is either a cycle or which has both endpoints in  $\partial M$ , then so does  $H$ .

## 7. UNTELESCOPING AND ESSENTIAL SURFACES

**Theorem 7.1.** *Let  $M$  be a compact, orientable 3-manifold containing a properly embedded graph  $T$  and let  $\Gamma$  be a subgraph of  $T$ . Furthermore assume that  $M - T$  is irreducible and that no sphere in  $M$  intersects  $T$  exactly once. Suppose  $H$  is a  $\Gamma$ -bridge surface for  $(M, T)$  that is  $(T, \Gamma)$ -c-weakly reducible and  $T$ -irreducible. Then there is a multiple  $\Gamma$ -bridge splitting for  $(M, T, \Gamma)$  such that*

- $\mathcal{H}^-$  is incompressible in the complement of  $T$ ;
- no component of  $\mathcal{H}^-$  is parallel to a component of  $\mathcal{H}^+$ ;
- each component of  $\mathcal{H}^+$  is  $(T, \Gamma)$ -c-strongly irreducible in  $M - \mathcal{H}^-$ ;
- and
- $\mathcal{H}$  is obtained from  $H$  by untelescoping and consolidation (possibly many times).

*Proof.* Untelescope  $H$  as described in Section 6.1 along a collection of  $(T, \Gamma)$ -c-disks. Let  $\mathcal{H}_1$  be the resulting  $\Gamma$ -bridge surface with a single (possibly disconnected) thin surface. We will define  $\mathcal{H}_i$  for  $i \geq 2$  inductively

as follows: If components of  $\mathcal{H}_{i-1}^-$  are parallel to components of  $\mathcal{H}_{i-1}^+$ , by Lemma 6.5, we may consolidate  $\mathcal{H}_{i-1}$  to obtain a multiple  $\Gamma$ -Heegaard splitting  $\mathcal{H}_i$  of  $(M, T, \Gamma)$ . If no component of  $\mathcal{H}_{i-1}^-$  is parallel to a component of  $\mathcal{H}_{i-1}^+$  but a component of  $\mathcal{H}_{i-1}^+$  is  $(T, \Gamma)$ -c-weakly reducible in the component of  $M - \mathcal{H}_{i-1}$  containing it, then we may untelescope  $\mathcal{H}_{i-1}$  to create  $\mathcal{H}_i$ .

Since both untelescoping and consolidation strictly reduce complexity, eventually this process terminates with a  $\Gamma$ -multiple Heegaard splitting  $\mathcal{H}$  of  $(M, T, \Gamma)$  such that:

- No component of  $\mathcal{H}^-$  is parallel to any component of  $\mathcal{H}^+$ ; and
- Each component of  $\mathcal{H}^+$  is  $(T, \Gamma)$ -c-strongly irreducible in the component of  $M - \mathcal{H}^-$  containing it.

By Lemma 6.1,  $\mathcal{H}^-$  is incompressible.  $\square$

We can now prove our main theorem.

**Theorem 7.2.** *Let  $M$  be a compact, orientable 3-manifold containing a properly embedded graph  $T$  and let  $\Gamma$  be a subgraph of  $T$ . Furthermore assume that  $M - T$  is irreducible and no sphere in  $M$  intersects  $T$  exactly once. Let  $H$  be a  $(T, \Gamma)$ -c-weakly reducible Heegaard surface for  $(M, T)$ . Then one of the following holds:*

- *there is a multiple  $\Gamma$ -Heegaard splitting  $\mathcal{H}$  for  $(M, T, \Gamma)$  so that each thin surface is  $T$ -essential and each thick surface is  $(T, \Gamma)$ -c-strongly irreducible in the component of  $M - \mathcal{H}^-$  containing it,*
- *$H$  contains a generalized stabilization,*
- *$H$  is perturbed, or*
- *$T$  has a removable path.*

*Proof.* Let  $\mathcal{H}$  be the multiple  $\Gamma$ -Heegaard splitting of  $(M, T, \Gamma)$  provided by Theorem 7.1.

Suppose that a component  $F$  of  $\mathcal{H}^-$  is  $T$ -parallel to  $\partial M$ . Let  $C_F$  be the compression body so that  $F = \partial_+ C_F$  is parallel to the boundary of a regular neighborhood of some components of  $\partial M$  together with some subset of  $T$ . By Lemma 3.4 we may assume that  $F$  is innermost, i.e.,  $C_F$  does not contain any other thin surfaces. Let  $H_{C_F}$  be the  $\Gamma$ -Heegaard splitting for  $C_F$  given by the unique thick surface of  $\mathcal{H}$  contained in  $C_F$ .

**Case 1:**  $T \cap C_F = \emptyset$ .

Recall that  $F$  is not parallel to  $H_{C_F}$ . If  $\partial_- C_F = \emptyset$ , then by [W],  $H$  is stabilized. If  $\partial_- C_F \neq \emptyset$ , by [ST2]  $H_{C_F}$  is stabilized or boundary stabilized along  $\partial_- C_F$ . See also [MS]. By Lemma 6.6,  $H$  contains a generalized stabilization.

**Case 2:**  $H_{C_F}$  is a Heegaard splitting.

Since no thin surface is parallel to a thick surface, by Theorem 3.6 one of the following occurs:

- $H_{C_F}$  has a generalized stabilization and if this is a boundary stabilization, it is along a component of  $\partial M$ ;
- $H_{C_F}$  is perturbed; or
- $T \cap C_F$  has a removable path disjoint from  $F$ .

By Lemma 6.6,  $H$  is either perturbed or it contains a generalized stabilization or a removable path as desired.

**Case 3:**  $H_{C_F}$  is a  $\Gamma$ -Heegaard splitting but not a Heegaard splitting.

Let  $A$  and  $B$  be the two compression bodies into which  $H_{C_F}$  divides  $C_F$ . Since  $H_{C_F}$  is a  $\Gamma$ -Heegaard splitting but not a Heegaard splitting, there exists an edge  $e \subset \Gamma$  in either  $A$  or  $B$  which is disjoint from  $H_{C_F}$  and which has both endpoints on  $\partial C_F$ .

**Case 3a:**  $\partial e \subset F$ .

Since  $F$  is  $T$ -parallel to  $\partial M \cup T$  and since  $e \subset T$  is an edge with both endpoints on  $F$ ,  $T = e$  and  $F = S^2$ . Then by [HS2, Lemma 2.1] and [HS1, Theorem 1.1], either  $H_{C_F}$  is stabilized, meridionally stabilized, or perturbed. See Case 2 of the proof of [T, Lemma 5.2] for details. By Lemma 6.6,  $H$  is stabilized, meridionally stabilized, or perturbed.

**Case 3b:**  $\partial e \subset \partial M$

In this case  $e$  is disjoint not only from  $H_{C_F}$  but also from  $H$ . Then  $e$  is an edge of  $\Gamma$  with both endpoints on  $\partial M$  which is disjoint from  $H$ , contrary to the hypothesis that  $H$  is a Heegaard surface.

**Case 3c:** One endpoint of  $e$  is on  $F$  and one endpoint of  $e$  is on  $\partial M$ .

We may assume that the hypotheses of cases (3a) and (3b) do not apply. Let  $e_1, \dots, e_n$  be the union of edges of  $T \cap C_F$  with one endpoint on  $F$ , one endpoint on  $\partial M$ , and which are disjoint from  $H_{C_F}$ . Perform a slight isotopy of each of them to convert  $T$  into a graph  $T'$  and each edge  $e_i$  into  $e'_i$  so that each  $e'_i$  is a removable edge of  $T'$  as in Lemma 2.2. Let  $H'_{C_F}$  be the new  $\Gamma$ -Heegaard surface and notice that  $H'_{C_F}$  is, in fact, a Heegaard surface for  $C_F$ . Since  $F$  is not parallel to  $H_{C_F}$ , by Theorem 3.6 one of the following occurs:

- $H'_{C_F}$  has a generalized stabilization and if this is a boundary stabilization, it is along a component of  $\partial M$ ;
- $H'_{C_F}$  is perturbed; or
- $T \cap C_F$  has a removable path with both endpoints in  $\partial M$ .

Notice that if the last option occurs the removable path is not equal to any of the  $e'_i$  since each of those edges has one endpoint on  $F$ . By Lemma 2.2 one of the following occurs:



- $H_{C_F}$  has a generalized stabilization and if this is a boundary stabilization, it is along a component of  $\partial M$ ;
- $H_{C_F}$  is perturbed; or
- $T \cap C_F$  has a removable path disjoint from  $F$ .

By Lemma 6.6, either  $H$  contains a generalized stabilization, or  $H$  is perturbed, or  $T$  contains a removable path.  $\square$

## 8. INTERSECTIONS BETWEEN A SURFACE AND A BRIDGE SURFACE

Let  $T$  be a finite trivalent graph containing at least one edge embedded in a closed orientable 3-manifold  $M$ . Let  $H$  be a Heegaard surface for  $(M, T)$ . We will consider the intersections between  $H$  and a surface  $F$  in the exterior of the graph. Throughout we let  $\Gamma = T$  and write “ $c$ -weakly reducible” for “ $T$ - $c$ -weakly reducible”, etc.

The result in this section will be used in the proof of Theorem 9.1; however, it is also of interest in its own right. The main idea of the proof can be traced back to Gabai’s proof of the Poenaru conjecture [G]. Similar ideas were also key to the results in [GST].

**Theorem 8.1.** *Suppose that  $F$  is a surface properly embedded in the exterior of  $T$ . Assume that  $\partial F$  is essential in  $\partial \eta(T)$  and that at least one component of  $\partial F$  is not a meridian of  $T$ . Then one of the following is true:*

- (1)  $H - T$  is  $c$ -weakly reducible in  $M - T$ .
- (2)  $M = S^3$ ,  $H = S^2$ , and  $T$  is the unknot in 1-bridge position with respect to  $H$ .
- (3)  $H$  is bimeridionally stabilized.
- (4)  $H$  is perturbed.
- (5)  $T$  has a perturbed level cycle and the associated cancelling pair of disks lie in  $F$ .
- (6)  $H$  can be isotoped so that  $F \cap H$  is a non-empty collection of arcs and circles and so that no arc or circle of intersection is inessential in  $F$ .

**8.1. Normal Form.** As in [GST, ST3, G] the main tool will be a sweepout of  $M$  by  $H$  and an examination of the upper and lower disks for  $T$  in  $F$ . We briefly recall the central concepts.

Let  $C_\uparrow$  and  $C_\downarrow$  be the handlebodies on either side of  $H$  in  $M$ . Removing spines for  $C_\uparrow$  and  $C_\downarrow$  from  $M$  creates a manifold homeomorphic to  $H \times (0, 1)$ . Let  $h: H \times (0, 1) \rightarrow (0, 1)$  be projection onto  $(0, 1)$  and extend  $h$  to be a height function  $h: M \rightarrow [0, 1]$ . The inverse image of  $t \in (0, 1)$  is a surface  $H_t$  isotopic to  $H$ . We choose the labelling of  $C_\uparrow$  and  $C_\downarrow$  so that for a specified  $H_t$ ,  $h^{-1}[t, 1)$  is  $C_\uparrow$  and  $h^{-1}(0, t]$  is  $C_\downarrow$ . That is,  $C_\uparrow$  lies above  $C_\downarrow$ . Isotope  $h$  so that  $T$  is in *normal form* with respect to  $h$  [S, Def. 5.1]. This means that

- (a) if  $e$  is an edge of  $T$ , the critical points of  $h|_e$  are nondegenerate and each lies in the interior of an edge.
- (b) the critical points of  $h|_{\text{edges}}$  and the vertices of  $T$  all occur at different heights, and
- (c) at each trivalent vertex  $v$  of  $T$  either two ends of incident edges lie above  $v$  (so that  $v$  is a  $y$ -vertex) or two ends of incident edges lie below  $v$  (so that  $v$  is a  $\lambda$ -vertex.)

$T$  can be perturbed by a small isotopy to be normal. We will always assume that  $T$  is in normal form with respect to  $h$ . The maxima of  $T$  consist of all local maxima of  $h|_{\text{edges}}$  and all  $\lambda$ -vertices. The minima of  $T$  consist of all local minima of  $h|_{\text{edges}}$  and all  $y$ -vertices. The *critical points* of  $T$  are all maxima and minima; the heights of the critical points are the *critical values*. Notice that since  $T$  is in bridge position with respect to  $H$ , all maxima are above all minima and we can interchange by an isotopy of  $T$  rel  $H$  the heights of two maxima or the heights of two minima.

If  $F \subset M - \mathring{\eta}(T)$  is a properly embedded surface,  $F$  is in *normal form* with respect to  $h$  [S, Def. 5.6] if

- (1) Each critical point of  $h$  on  $F$  is non-degenerate;
- (2) Each component of  $\partial F$  on  $\partial \eta(T)$  is either a horizontal meridional circle or contains only non-degenerate critical points, and occurs near an associated critical point of  $T$  in  $\partial \eta(T)$ ; the number of critical points has been minimized up to isotopy;
- (3) No critical point of  $h$  on  $\mathring{F}$  occurs near a critical height of  $h$  on  $T$ ;
- (4) No two critical points of  $h$  on  $\mathring{F}$  or  $\partial F$  occur at the same height;
- (5) The minima (resp. maxima) of  $h|_{\partial F}$  at the minima (resp. maxima) of  $T$  are half-center singularities; and
- (6) The maxima of  $h|_{\partial F}$  at  $y$  and  $\lambda$ -vertices are half-saddle singularities of  $h$  on  $F$ .

The surface  $F$  can always be properly isotoped to be in normal form.

**8.2. Bridge and loop boundary compressing disks and their associated cut and compressing disks.** Suppose that  $t \in (0, 1)$  and that  $D$  is a compressing disk or  $\partial$  compressing disk for  $H_t - \mathring{\eta}(T)$  in  $M - T$ . If  $D$  lies above  $H_t$ , we call  $D$  an *upper disk* and if  $D$  lies below  $H_t$ , we call  $D$  a *lower disk*. For certain values of  $t$ , an upper or lower disk has a specified form. Let  $t_{\min}$  be the height of the highest minimum of  $h|_T$  and let  $t_{\max}$  be the height of the lowest maximum of  $h|_T$ .

Suppose that  $D$  is an upper or lower disk for  $H_t$  with  $t \in (t_{\min}, t_{\max})$ . If  $D$  is a  $\partial$ -compressing disk for  $H - T$  such that the arc  $\partial D \cap H$  has endpoints in distinct components of  $\partial(H - \mathring{\eta}(T))$  then  $D$  is called a *bridge boundary*

*compressing disk*. See Figure 2. If the endpoints of  $\partial(D \cap H)$  lie in the same component of  $\partial(H - \mathring{\eta}(T))$ , then  $D$  is a *loop boundary compressing disk*.

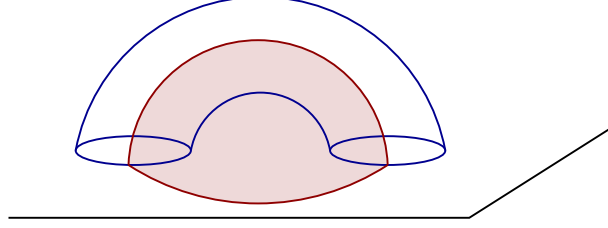


FIGURE 2. An example of a bridge boundary compression

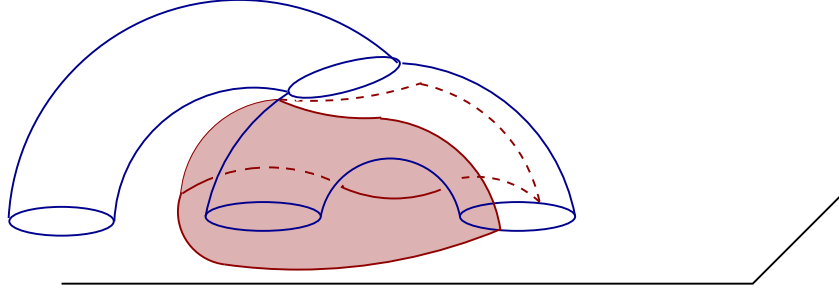


FIGURE 3. An example of a loop boundary compression

If  $D$  is a  $\partial$ -compressing disk for  $H - \mathring{\eta}(T)$ ,  $\partial D$  can be extended so that the interior of  $D$  lies in  $M - T$  and  $\partial D$  consists of an arc in  $H$  and an arc in  $T$ . We let  $\overline{D}$  denote the extended disk. If  $D$  is a bridge boundary compressing disk, then the frontier of a regular neighborhood of  $\overline{D}$  in  $C_\uparrow$  or  $C_\downarrow$  is a disk  $D'$  which has boundary in  $H - T$  and intersects  $T$  in 0 or 1 points. If  $D'$  is disjoint from  $T$ , then either  $D'$  is a compressing disk for  $H - T$  or  $H = S^2$ ,  $|T \cap H| = 2$ , and  $T$  is the unknot in  $M = S^3$ . If  $|D' \cap T| = 1$ , then either  $D'$  is a cut disk for  $H - T$ , or  $H = S^2$ ,  $|T \cap H| = 3$ , and  $T$  is a connected graph with two vertices and three edges. If  $D'$  is a compressing disk or cut disk for  $H - T$ , then we say that  $D'$  is *associated* to  $D$  and  $\overline{D}$ .

If  $D$  is a loop boundary compressing disk,  $\partial \overline{D}$  consists of a loop in  $H$ . The frontier of a regular neighborhood of  $\overline{D}$  consists of two disks,  $D'_1$  and  $D'_2$ . Each of  $D'_1$  and  $D'_2$  intersects  $T$  exactly once.

**Lemma 8.2.** *If neither  $D'_1$  nor  $D'_2$  is a cut disk, then  $H = S^2$  and  $|T \cap H| = 3$ .*

*Proof.* Without loss of generality, suppose that  $D \subset C_\uparrow$ . Since neither  $D'_1$  nor  $D'_2$  is a cut disk each of their boundaries must be parallel in  $H - \mathring{\eta}(T)$  to a component of  $\partial(H - \mathring{\eta}(T))$ . In fact, if  $D'_j$  intersects an edge  $e$  of  $T \cap C_\uparrow$ ,

then  $\partial D'_j$  is isotopic to  $\partial \eta(e) \cap H$ . By construction,  $D'_1$  and  $D'_2$  intersect distinct edges of  $T \cap C_\uparrow$ . Thus,  $H - (\partial D'_1 \cup \partial D'_2)$  consists of three components: two disks, each once punctured by  $T$ , and an annulus  $A$  punctured once by  $T$ . (The puncture  $T \cap A$  is the point  $T \cap \partial \bar{D} \cap H$ .) Thus,  $H = S^2$ , and  $|T \cap H| = 3$ .  $\square$

**Lemma 8.3.** *Suppose that  $D'_1$  is not a cut disk for  $H - T$  and that  $D \subset C_\uparrow$ . Then there exists a bridge disk  $E$  for  $T \cap C_\uparrow$  such that the edge of  $T \cap C_\uparrow$  lying in  $\bar{D}$  is contained in  $\partial E$  and any properly embedded arc in  $H - T$  which has been isotoped to intersect  $\partial E \cup \partial \bar{D}$  minimally intersects  $\partial E$  if and only if it intersects  $\partial \bar{D}$ .*

*Proof.* Since  $D'_1$  is not a cut disk and since  $T$  is in bridge position with respect to  $H$ , there exists a 3-ball  $B$  embedded in  $C_\uparrow$  such that  $\partial B = D'_1 \cup E'$  where  $E'$  is a disk in  $H$  punctured once by  $T$ . Let  $e$  be the edge of  $T \cap C_\uparrow$  which lies in  $\bar{D}$ . The ball  $B$  can be extended to a ball  $B'$  such that  $\bar{D} \subset \partial B'$ ,  $\partial B' \cap H$  is a disk punctured once by  $T$ , and the interior of  $B'$  contains an edge  $e'$  of  $T \cap C_\uparrow$ . Since  $T$  is in bridge position with respect to  $H$ , there exists a bridge disk  $E$  for  $T \cap C_\uparrow$  such that  $\partial E \cap T = e \cup e'$  and  $E \subset B'$ . Since  $B' \cap (H - T)$  is a once-punctured disk and since  $\partial E \cap H$  joins  $\partial(B' \cap H)$  to  $T \cap B' \cap H$ , any arc in  $B' \cap (H - T)$  which intersects both  $\partial E$  and  $\partial \bar{D}$  minimally intersects one if and only if it intersects the other.  $\square$

**Proof of Theorem 8.1.** Assume that  $H - T$  is c-strongly irreducible in  $M - T$  and that (2) does not occur.

**Claim 1:** If  $H = S^2$  and if  $|T \cap H| = 3$ , then  $T$  is perturbed.

*Proof:* Let  $v_1, v_2$ , and  $v_3$  be the points of  $T \cap H$ . Let  $D_\uparrow$  be a bridge disk for  $T \cap C_\uparrow$  so that  $D_\uparrow \cap H$  is an arc which joins  $v_1$  and  $v_2$ . Let  $D_\downarrow$  be a bridge disk for  $T \cap C_\downarrow$  so that  $D_\downarrow \cap H$  is an arc which joins  $v_2$  and  $v_3$ . Since  $H - T$  is a thrice-punctured sphere, the disks  $D_\uparrow$  and  $D_\downarrow$  can be isotoped in  $H - T$  so that the arcs  $D_\uparrow \cap H$  and  $D_\downarrow \cap H$  have disjoint interiors in  $H - T$ . Then  $\{D_\uparrow, D_\downarrow\}$  is a perturbing pair for  $H$ .  $\square$ (Claim 1)

Since the conclusion that  $T$  is perturbed is one of the possible conclusions of Theorem 8.1, we assume from now on that if  $H = S^2$ , then  $|T \cap H| \neq 3$ .

**Claim 2:** Suppose that  $t \in (t_{\min}, t_{\max})$ . Then after an isotopy of  $H$  to eliminate arcs and circles of intersection of  $H \cap F$  which are inessential in both  $H - \hat{\eta}(T)$  and  $F$ , there is not simultaneously an upper disk and a lower disk for  $H_t - \hat{\eta}(T)$  such that both disks lie in  $F$ .

*Proof:* Suppose that there is an upper disk  $D_\uparrow$  and lower disk  $D_\downarrow$  for  $H_t - \hat{\eta}(T)$ , such that both disks lie in  $F$ . If  $t$  is a critical value of  $h|_F$ , the interiors of  $\partial D_\uparrow \cap H$  and  $\partial D_\downarrow \cap H$  may not be disjoint. In this case, however, a small isotopy of one of them will make the interior of  $\partial D_\uparrow \cap H$  disjoint

from the interior of  $\partial D_\downarrow \cap H$ . We will always assume that this isotopy, if needed, has been performed. If  $D_\uparrow$  or  $D_\downarrow$  is a  $\partial$ -compressing disk, extend it to a disk  $\overline{D}_\uparrow$  or  $\overline{D}_\downarrow$  as before.

As the situation is symmetric with respect to  $\uparrow$  and  $\downarrow$  and there are three possibilities for each kind of disk: compressing, bridge  $\partial$ -compressing and loop  $\partial$ -compressing. There are six cases to consider; we will handle two of them: one disk a compressing disk and the other either kind of  $\partial$ -compressing disk, at the same time.

**Case 1:** Both  $D_\uparrow$  and  $D_\downarrow$  are compressing disks.

We may assume that  $D_\uparrow$  and  $D_\downarrow$  are disjoint. (If they were not a small isotopy, as above, would make them so.) Thus  $H - T$  is c-weakly reducible, a contradiction.

**Case 2:** One disk, say  $D_\uparrow$ , is a compressing disk and  $D_\downarrow$  is a  $\partial$ -compressing disk.

Since  $D_\uparrow$  is a compressing disk, if  $H = S^2$ , then  $|T \cap H| \geq 4$ . Thus, if  $D_\downarrow$  is a bridge boundary compressing disk, there is a c-disk  $D'_\downarrow$  for  $H - T$  associated to  $\overline{D}_\downarrow$  and if  $D_\downarrow$  is a loop boundary compressing disk, there is an associated cut-disk  $D'_\downarrow$ . In either case the boundaries of  $D_\uparrow$  and  $D_\downarrow$  are disjoint, so  $D_\uparrow \cap D'_\downarrow = \emptyset$  and so  $H - T$  is c-weakly reducible, a contradiction.

**Case 3:** Both  $D_\uparrow$  and  $D_\downarrow$  are bridge boundary compressing disks.

The arcs  $\overline{D}_\uparrow \cap H$  and  $\overline{D}_\downarrow \cap H$  have disjoint interiors. If they also have disjoint endpoints, then  $|T \cap H| \geq 4$  and there are disjoint c-disks  $D'_\uparrow$  and  $D'_\downarrow$  associated to  $D_\uparrow$  and  $D_\downarrow$ , respectively. In this case,  $H - T$  is c-weakly reducible contradicting our assumption. If  $\overline{D}_\uparrow \cap H$  and  $\overline{D}_\downarrow \cap H$  share exactly one endpoint then  $H$  is perturbed, so assume that they share both endpoints. In this case, the cycle in  $T$  which is the closure of  $(\partial \overline{D}_\uparrow \cap \partial \overline{D}_\downarrow) - H$ , is a perturbed level cycle. Thus,  $T$  has a perturbed level cycle and the associated cancelling pair of disks lie in  $F$ .

**Case 4:** One of the disks, say  $D_\uparrow$ , is a bridge boundary compressing disk and  $D_\downarrow$  is a loop boundary compressing disk.

Suppose, first, that  $\partial \overline{D}_\uparrow$  and  $\partial \overline{D}_\downarrow$  are disjoint. Let  $D'_\uparrow$  be the c-disk associated to  $\overline{D}_\uparrow$ . Let  $D_\downarrow^1$  and  $D_\downarrow^2$  be the once-punctured disks associated to  $\overline{D}_\downarrow$ . If  $H = S^2$ , then  $|T \cap H| \neq 3$ , so we may assume that  $D_\downarrow^1$ , say, is a cut disk for  $H - T$ . Since  $\overline{D}_\uparrow$  and  $\overline{D}_\downarrow$  are disjoint,  $D'_\uparrow$  and  $D_\downarrow^1$  are disjoint and so  $H - T$  is c-weakly reducible, a contradiction.

Thus, we may assume that  $\partial \overline{D}_\uparrow \cap H$  and  $\partial \overline{D}_\downarrow \cap H$  intersect. The intersection must be one of the endpoints of the arc  $\partial \overline{D}_\uparrow \cap H$ ; let  $v$  be this point of  $T \cap H$ . One of  $D_\downarrow^1$  or  $D_\downarrow^2$  is disjoint from  $\overline{D}_\uparrow$ . Without loss of generality, suppose it to be  $D_\downarrow^1$ . If  $D_\downarrow^1$  is a cut disk, then the disks  $D'_\uparrow$  and  $D_\downarrow^1$  show that

$H - T$  is c-weakly reducible, a contradiction. Thus,  $D_\downarrow^1$  is not a cut disk. Let  $E_\downarrow$  be the bridge disk provided by Lemma 8.3. Since  $D_\downarrow^1$  is disjoint from  $\overline{D}_\uparrow$ , the interior of the arc  $\partial E_\downarrow \cap H$  is disjoint from the interior of the arc  $\overline{D}_\uparrow$ . Notice that, by the construction of  $E$  from  $D_\downarrow^1$ , the arcs  $\partial E_\downarrow \cap H$  and  $\partial \overline{D}_\uparrow \cap H$  share exactly one endpoint. Thus,  $T$  is perturbed.

**Case 5:** Both  $D_\uparrow$  and  $D_\downarrow$  are loop boundary compressing disks.

Let  $D_j^1, D_j^2$  be the once-punctured disks associated to  $D_j$  for  $j \in \{\uparrow, \downarrow\}$ . Suppose, first, that  $\overline{D}_\uparrow$  and  $\overline{D}_\downarrow$  are disjoint. If  $H = S^2$ , then  $|H \cap T| \neq 3$ . Thus, one of  $D_j^1$  and  $D_j^2$  is a cut disk for  $j \in \{\uparrow, \downarrow\}$ . Without loss of generality, suppose that both  $D_\uparrow^1$  and  $D_\downarrow^1$  are cut disks. Since  $\overline{D}_\uparrow$  and  $\overline{D}_\downarrow$  are disjoint, the cut disks  $D_\uparrow^1$  and  $D_\downarrow^1$  are disjoint. Thus,  $H - T$  is c-weakly reducible, a contradiction.

We may, therefore, assume that  $\overline{D}_\uparrow$  and  $\overline{D}_\downarrow$  intersect at a single point  $v \in T \cap H$ .

**Case 5a:** The endpoints of  $\partial D_\uparrow$  on  $\partial \eta(v) \subset H$  do not alternate with the endpoints of  $\partial D_\downarrow$ .

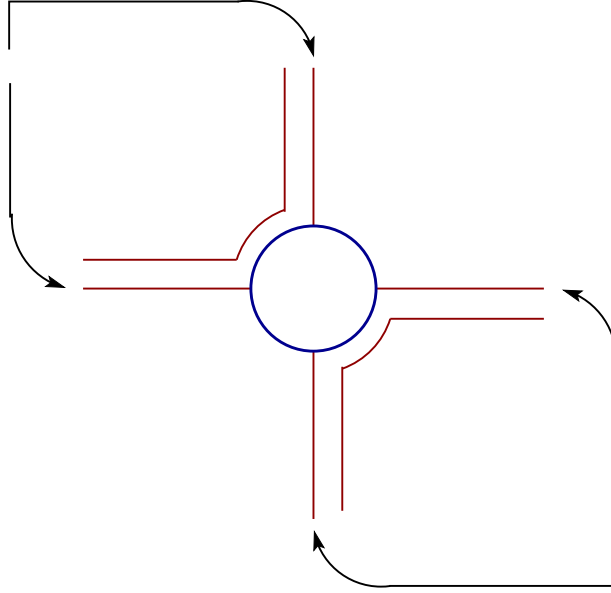
Choose the labelling of  $D_\uparrow^1, D_\uparrow^2$  and  $D_\downarrow^1, D_\downarrow^2$  carefully so that the portion of  $\partial D_\uparrow^1$  parallel to  $\partial \eta(T \cap H)$  and the portion of  $\partial D_\downarrow^1$  parallel to  $\partial \eta(T \cap H)$  are on opposite sides of  $\partial \eta(T \cap H)$ . See Figure 4. A small isotopy of  $D_\downarrow^1$  will also guarantee that  $D_\uparrow^2$  and  $D_\downarrow^1$  are disjoint. By claim 1 we may assume that if  $H = S^2$  then  $|T \cap H| \neq 3$ , and so one of  $D_\uparrow^1$  or  $D_\uparrow^2$  is a cut disk for  $H - T$ . It follows that if  $D_\downarrow^1$  is a cut disk for  $H - T$ , then  $H - T$  is c-weakly reducible contradicting our assumption. We conclude that  $D_\uparrow^2$  is a cut disk for  $H - T$ . A small isotopy makes  $D_\uparrow^1$  disjoint from  $D_\downarrow^2$ . Thus, if  $D_\uparrow^1$  is a cut disk the disks  $D_\uparrow^1$  and  $D_\downarrow^2$  show that  $H - T$  is c-weakly reducible, a contradiction. Thus, neither  $D_\uparrow^1$  nor  $D_\downarrow^1$  is a cut disk. Let  $E_\uparrow$  and  $E_\downarrow$  be the disks obtained by applying Lemma 8.3 to  $D_\uparrow^1$  and  $D_\downarrow^1$  respectively. By our choice of labelling for  $D_\uparrow^1$  and  $D_\downarrow^1$ , the arcs  $E_\uparrow \cap H$  and  $E_\downarrow \cap H$  share a single endpoint. Hence,  $H$  is perturbed.

**Case 5b:** The endpoints of  $\partial D_\uparrow$  on  $\partial \eta(v) \subset H$  alternate with the endpoints of  $\partial D_\downarrow$ .

In this case, each of the pairs  $\{D_\uparrow^j, D_\downarrow^k\}$  for  $\{j, k\} \subset \{1, 2\}$  is a pair of once-punctured disks with boundaries intersecting transversally exactly once and so  $H$  is bimeridionally stabilized.

□(Claim 2)




 FIGURE 4. Choosing the labels  $D_{\uparrow}^1$  and  $D_{\downarrow}^1$ 

Interchange maxima and interchange minima of  $h|_T$  as necessary to guarantee that  $h|_F$  has critical values at  $t_{\min}$  and  $t_{\max}$ . Let

$$t_{\min} = s_0 < s_1 < \dots < s_n = t_{\max}$$

be the critical values of  $h|_F$  between  $t_{\min}$  and  $t_{\max}$ . Let  $I_i = (s_i, s_{i+1})$  for  $1 \leq i \leq n-1$ . Label  $I_i$  with  $\uparrow$  if for  $t \in I_i$ , there exists an upper disk for  $H_t$  in  $F$  and label  $I_i$  with  $\downarrow$  if there exists a lower disk for  $H_t$  in  $F$ . By Claim 2, no  $I_i$  is labelled both  $\uparrow$  and  $\downarrow$ . Furthermore, for no  $i$  is  $I_i$  labelled  $\downarrow$  and  $I_{i+1}$  labelled  $\uparrow$ , since  $H_{t_{i+1}}$  would have both an upper disk and a lower disk contained in  $F$ . Since  $h|_F$  has critical values at  $t_{\min}$  and  $t_{\max}$  and since  $F$  is in normal form,  $I_0$  is labelled  $\downarrow$  and  $I_{n-1}$  is labelled  $\uparrow$ . Thus, there exists some  $i$  so that  $I_i$  is not labelled  $\uparrow$  or  $\downarrow$ . Isotope  $H_t$  to eliminate arcs and circles of intersection which are inessential on both  $H_t - \mathring{\eta}(T)$  and  $F$ . By our choice of  $t_{\min}$  and  $t_{\max}$ , for  $t \in I_i$ ,  $H_t \cap F$  is non-empty. Since  $I_i$  is unlabelled, each arc and circle of intersection is essential in  $F$ .  $\square$ (Theorem)

## 9. LEVELING EDGES

Suppose  $M$  is a closed manifold. A *Heegaard spine* for  $M$  is a graph  $T \subset M$  such that the exterior of  $T$  in  $M$  is a handlebody. The *genus* of  $T$  is defined to be the genus of  $\partial\eta(T)$ . We say that  $T$  is *reducible* if there exists a sphere in  $M$  intersecting an edge of  $T$  transversally in a single point. If  $T$  is not reducible, it is *irreducible*.

In this section we will prove the following.

**Theorem 9.1.** *Suppose  $H$  is a Heegaard surface for  $M$  and  $T$  is an irreducible trivalent Heegaard spine for a closed manifold  $M$  in minimal bridge position with respect to  $H$ . Then one of the following occurs:*

- (1)  $H$  is stabilized, meridionally stabilized, or bimeridionally stabilized as a splitting of  $(M, T)$ .
- (2)  $T$  has a perturbed level edge.
- (3)  $T$  contains a perturbed level cycle.
- (4) There is an essential meridional surface  $F$  in the exterior of  $T$  such that  $\text{genus}(F) \leq \text{genus}(H)$ .

This theorem is a partial generalization of the following result of Scharle-mann and Thompson:

**Theorem 9.2.** [ST3] *Suppose that  $T$  is a trivalent genus 2 Heegaard spine for  $S^3$ . If  $T$  is isotoped to be in thin position with respect to a Heegaard sphere  $H$  for  $S^3$  then  $T$  is in extended bridge position with respect to  $H$  and some interior edge of  $T$  is a perturbed level edge.*

We begin by proving a corollary of Theorem 9.1. For the proof we will need the following result of Morimoto:

**Theorem 9.3.** [M] *Suppose that  $T$  is a genus 2 Heegaard spine for a closed orientable 3-manifold  $M$ . Suppose that  $S \subset M$  is a 2-sphere transverse to  $T$  such that  $S \cap T$  is contained in non-separating edges of  $T$ . Then either  $M$  contains a lens space or  $S^1 \times S^2$  connected summand or  $S - \mathring{\eta}(T)$  is inessential in  $M - \mathring{\eta}(T)$ .*

**Corollary 9.4.** *Suppose that  $T$  is a genus 2 trivalent Heegaard spine for  $S^3$  in minimal bridge position with respect to a Heegaard sphere  $H$  for  $S^3$ . Suppose that every edge of  $T$  is non-separating in  $T$ . Then some edge  $e$  of  $T$  can be isotoped to lie in  $H$  by a proper isotopy in  $M - \mathring{\eta}(T - e)$ .*

*Proof.* Since  $H$  is not stabilized as a splitting of  $S^3$ , it is neither stabilized nor meridionally stabilized as a splitting of  $(S^3, T)$ . Since every edge of  $T$  is non-separating and  $\text{genus}(T) \geq 2$ , no edge transversally intersects a sphere in  $S^3$  exactly once. By Morimoto's theorem, the exterior of  $T$  does not contain an essential meridional planar surface. Thus, by Theorem 9.1,  $T$  contains a perturbed level edge or a perturbed level cycle. Let  $e$  be either the perturbed level edge, or one of the edges contained in the perturbed level cycle. It is not difficult to use the perturbing disks to isotope  $e$  in  $M - \mathring{\eta}(T - e)$  so that it lies in  $H$ .  $\square$

For the remainder we consider the following more general situation. Let  $T$  be a finite trivalent graph containing at least one edge embedded in a

compact orientable 3-manifold  $M$ . For simplicity, assume that  $T \cap \partial M = \emptyset$ . Let  $H$  be a Heegaard surface for  $(M, T)$  dividing  $M$  into compression bodies  $C_1$  and  $C_2$ .

We begin by considering the various methods of unperturbing Heegaard surfaces. Let  $D_1 \subset C_1$  and  $D_2 \subset C_2$  be disks which form a perturbing pair for  $H$ . Let  $p = \partial D_1 \cap \partial D_2$ .

The next lemma is essentially [STo, Lemma 3.1].

**Lemma 9.5.** *If  $\partial D_1 \cup \partial D_2$  is disjoint from the vertices of  $T$ , then there is an isotopy of  $T$  which reduces  $|T \cap H|$  by two and which is supported in a regular neighborhood of  $D_1 \cup D_2$ .*

This lemma is stated, but not proved, in [HS1, Section 3].

**Lemma 9.6.** *Suppose that  $\partial D_1$  contains one vertex  $v$  of  $T$  and that  $\partial D_2$  does not contain any vertices of  $T$ . Let  $\tau$  be the component of  $T \cap C_2$  adjacent to  $(\partial D_1 \cap (T \cap H)) - p$ . If  $\tau$  does not contain a vertex of  $T$ , then there is an isotopy of  $T$  supported in a neighborhood of  $D_1 \cup D_2$  which reduces  $|T \cap H|$ .*

*Proof.* Choose a complete collection of bridge disks  $\Delta$  for  $T \cap C_2$  containing  $D_2$ . Out of all such collections, choose  $\Delta$  to minimize  $|\partial \Delta \cap (\partial D_1 \cap H)|$ .

**Claim:**  $\partial \Delta$  is disjoint from the interior of  $\partial D_1 \cap H$ .

Suppose not, and let  $x$  be the point of  $\partial \Delta \cap \text{int}(\partial D_1 \cap H)$  closest to  $p$ . Let  $\alpha$  be the path in  $\partial D_1 \cap H$  from  $x$  to  $p$ . Let  $D$  be the disk of  $\Delta$  containing  $x$ . The frontier in  $C_2$  of a regular neighborhood of  $D \cup \alpha \cup D_2$  has a disk component  $D'$  which is a bridge disk for  $\partial D \cap T \cap C_2$ . Replacing  $D$  with  $D'$  in  $\Delta$  creates a collection which contradicts our choice of  $\Delta$ .  $\square$ (Claim)

Let  $\tau'$  be the component of  $T \cap C_1$  which contains  $v$  and let  $e_1$  be the edge of  $\tau'$  which lies in  $\partial D_1$  but does not contain  $p$ . Let  $e$  be  $(\partial D_1 \cup \partial D_2) - e_1$ . Isotope  $\tau'$  so that  $e_2 = e \cap C_1$  is moved across  $D_1$  into  $C_2$ . See Figure 5. Let  $S$  be the resulting parallelism between  $e_2$  and  $H$  in  $C_2$ . By the Claim,  $\Delta - D_2$  is disjoint from  $S$ . Thus,  $(\Delta - D_2) \cup (D_2 \cup S)$  is a complete collection of bridge disks for  $T \cap C_2$ . Since  $\tau$  does not contain a vertex of  $T$ , each component of  $T \cap C_2$  is a pod or a bridge edge. Thus,  $H$  is still a Heegaard splitting of  $(M, T)$ .

$\square$

**Corollary 9.7.** *Suppose that  $T \cap \partial M = \emptyset$  and suppose that  $T$  has been isotoped so that it is in minimal bridge position with respect to  $H$ . If  $T$  is perturbed, it contains a perturbed level edge.*

*Proof.* Suppose that  $H$  is perturbed by disks  $D_1 \subset C_1$  and  $D_2 \subset C_2$ . By Lemma 9.5, one of  $\partial D_1$  or  $\partial D_2$  contains a vertex of  $T$ . If both do, then  $T$  has

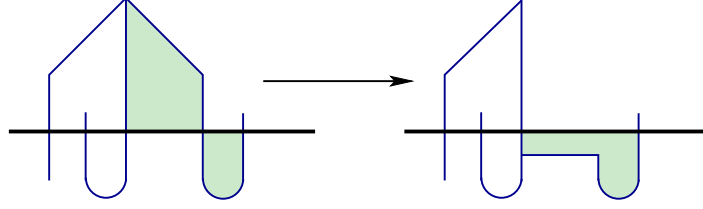


FIGURE 5. Unperturbing a Heegaard splitting

a perturbed level edge. Suppose, therefore, that only  $\partial D_1$  contains a vertex  $v$  of  $T$ . Let  $\tau$  be the component of  $T \cap C_2$  adjacent to  $(\partial D_1 \cap T \cap H) - p$ . By Lemma 9.6,  $\tau$  contains a vertex  $w$  of  $T$ . There is, therefore, an edge  $e$  joining  $v$  to  $w$  and intersecting  $H$  exactly once. Since  $T \cap \partial M = \emptyset$ , there is an edge  $e' \subset C_2$  adjacent to  $w$  which is disjoint from  $\partial D_1 \cup \partial D_2$ . Let  $D'$  be a bridge disk for  $e' \cup (e \cap C_2)$ . By the proof of Lemma 9.6, we may assume that the arc  $\partial D' \cap H$  has its interior disjoint from the arc  $\partial D_1 \cap H$ . Thus,  $(D_1, D')$  is a perturbing pair for  $T$  which shows that  $e$  is a perturbed level edge.  $\square$

**Proof of Theorem 9.1.** Let  $T$  be an irreducible trivalent Heegaard spine for the closed 3-manifold  $M$ . Let  $F$  be a complete collection of boundary reducing disks for the exterior of  $T$ . Since  $T$  is irreducible, no component of  $\partial F$  is a meridian of  $T$ .

Isotope  $T$  so that it is in minimal bridge position with respect to  $H$ . Assume that neither conclusion (1) nor conclusion (4) occurs. If  $H - T$  is c-weakly reducible, by Theorem 7.2, either  $H$  is perturbed or  $T$  contains a perturbed level cycle. If  $H - T$  is c-strongly irreducible in  $M - T$ , then by Theorem 8.1, one of the following occurs:

- (a)  $H$  is perturbed
- (b)  $T$  has a perturbed level cycle
- (c)  $H$  can be isotoped to intersect  $F$  in a non-empty collection of arcs and simple closed curves all of which are essential in  $F$ .

Since  $F$  is the pairwise disjoint union of disks, (c) is impossible. If  $H$  is perturbed, by Corollary 9.7,  $T$  has a perturbed level edge. Thus,  $T$  has either a perturbed level edge or a perturbed level cycle.  $\square$

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